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GLOBAL EXISTENCE AND ASYMPTOTIC STABILITY FOR A NONLINEAR INTEGRODIFFERENTIAL EQUATION MODELING HEAT FLOW

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ABSTRACT

We study initial-value problems that arise from models for one-dimensional heat flow (with finite wave speeds) in materials with memory. Under assumptions that ensure compatibility of our constitutive relations with the second law of thermodynamics, the resulting integrodifferential equation is hyperbolic near equilibrium. We establish the existence of unique, global (in time) defined, classical solutions to the problems under consideration, provided the data are smooth and sufficiently close to equilibrium. We treat both Dirichlet and Neumann boundary conditions as well as the problem on the entire real line.

Local existence is proved using a contraction-mapping argument which involves estimates for linear hyperbolic PDE's with variable coefficients. Global existence is obtained by deriving a priori energy estimates. These estimates are based on inequalities for strongly positive Volterra kernels (including a new inequality that is needed due to the form of the constitutive relations). Furthermore, compatibility with the second law plays an essential role in the proof in order to obtain an existence result under less restrictive assumptions on the data.

AMS (MOS) Subject Classifications: 45K05, 35L60, 80A20.

Key Words: Integrodifferential equation, second sound, heat flow, hyperbolic equation.

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0. Introduction

In this paper we establish global existence and asymptotic stability of solutions to initial-value problems arising from integral models for heat flow that were introduced in [2]. These models are based on Gurtin and Pipkin's theory of heat conduction [6]. The situations we are concerned with are such that the heat flux depends on the temporal history of the temperature gradient (and possibly on the present value and the history of the temperature), but is independent of the present value of the temperature gradient.

As in [2], we restrict our attention to one-dimensional rigid heat conductors in which the only nonzero component of the heat flux is its x-component, q. Here q and the absolute temperature $\theta > 0$ are functions of x and time t. Moreover, we assume that the material under consideration is homogeneous and has unit density. The first two laws of thermodynamics then take the form

$$e_t + q_x = r, \tag{0.1}$$

$$\eta_t \geq -(\frac{q}{\theta})_x + \frac{r}{\theta}, \tag{0.2}$$

where $e = e(x,t)$ is the (specific) internal energy, $r = r(x,t)$ is the external heat supply, and $\eta = \eta(x,t)$ is the (specific) entropy. Subscripts t and x indicate partial derivatives. If we define the (specific) free energy $\psi = \psi(x,t)$ through

$$\psi := e - \theta\eta \tag{0.3}$$

then the law of balance of energy (0.1) and the entropy inequality (0.2) can be combined to give the Clausius-Duhem inequality

$$\psi_t + \eta\theta_t + \frac{q\theta_x}{\theta} \leq 0. \tag{0.4}$$

Gurtin and Pipkin consider materials characterized by constitutive equations that express $\psi(x,t)$, $\eta(x,t)$, and $q(x,t)$ as functionals of $(\theta(x,t), \bar{\theta}^t(x,\cdot), \bar{\theta}^t_x(x,\cdot))$. Here $\bar{\theta}^t$ and $\bar{\theta}^t_x$ denote the summed histories up to time t of the temperature and the temperature gradient. The summed history up to time t of θ is defined by

$$\bar{\theta}^t(x,s) := \int_{t-s}^{t} \theta(x,z)dz \qquad \forall x \epsilon B,\ s \geq 0, \tag{0.5}$$

where $B \subset \mathbb{R}$ denotes the interval occupied by the body. Gurtin and Pipkin require that their constitutive relations be compatible with thermodynamics in the sense that the Clausius-Duhem inequality (0.4) is satisfied for all smooth processes consistent with the constitutive relations. They derive conditions that are both necessary and sufficient for compatibility with thermodynamics. These conditions can be summarized roughly as follows:

(i) the entropy is minus the derivative of the free energy with respect to the present value of the temperature;

(ii) the heat flux is determined from the free energy through a differential equation called the heat flux relation;

(iii) a functional differential inequality called the dissipation inequality holds for all smooth processes.

We note that by virtue of (0.3), condition (ii) implies a relation between q and e and hence, e will generally depend on $\bar{\theta}^t_x$.

MacCamy considered a model motivated by Gurtin and Pipkin's linearized constitutive equations [10]. He replaced the linear equation for the heat flux with

$$q(x,t) = -\int_0^\infty a(s)f(\theta_x(x,t-s))ds, \tag{0.6}$$

but retained the linear equation for the internal energy

$$
\begin{aligned}
e(x,t) &= b + c\theta(x,t) - \int_0^\infty \beta'(s)\bar{\theta}^t(x,s)ds \\
&= b + c\theta(x,t) + \int_0^\infty \beta(s)\theta(x,t-s)ds.
\end{aligned}
\tag{0.7}
$$

Here b and c are constants, a and β are smooth kernels that decay sufficiently rapidly at infinity, and f is a smooth function. MacCamy proved global existence and asymptotic stability for a corresponding initial/boundary-value problem. Similar existence theorems for MacCamy's model were established by Dafermos and Nohel [5] and Staffans [12].

MacCamy does not address the issue of compatibility with thermodynamics. However, one can show that there are smooth processes consistent with (0.6), (0.7) but for which an inequality implied by (0.4) is violated; within the context of [5], [10], and [12] this probably is not a serious difficulty since the solutions discussed there remain close to equilibrium (i.e. close to a state where θ is a constant and $\theta_x \equiv 0$), and under reasonable assumptions on a, β, and f the aforementioned inequality is satisfied by a suitable class of processes that are close to equilibrium. (See Section 1 of [2] for further details.)

Here we consider the constitutive relations[1]

$$
\begin{aligned}
\psi(x,t) &= \hat{\psi}(\theta(x,t)) + \int_0^\infty \hat{\Psi}(s,\theta(x,t),\,\bar{\theta}^t(x,s),\,\bar{\theta}_x^t(x,s))ds, \\
\eta(x,t) &= -\hat{\psi}'(\theta(x,t)) - \int_0^\infty \hat{\Psi}_{,2}(s,\theta(x,t),\,\bar{\theta}^t(x,s),\,\bar{\theta}_x^t(x,s))ds, \\
q(x,t) &= -\theta(x,t)\int_0^\infty \hat{\Psi}_{,4}(s,\theta(x,t),\,\bar{\theta}^t(x,s),\,\bar{\theta}_x^t(x,s))ds,
\end{aligned}
\tag{0.8}
$$

and hence by (0.3) we have

$$
e(x,t) = \hat{e}(\theta(x,t)) + \int_0^\infty \hat{E}(s,\theta(x,t),\,\bar{\theta}^t(x,s),\,\bar{\theta}_x^t(x,s))ds \tag{0.9}
$$

[1] We use $F_{,j}$ to denote the partial derivative of a function F with respect to its j-th argument.

with

$$\hat{e}(\nu) := \hat{\psi}(\nu) - \nu\hat{\psi}'(\nu),\ \hat{E}(s,\nu,\alpha,\gamma) := \hat{\Psi}(s,\nu,\alpha,\gamma) - \nu\hat{\Psi},_2(s,\nu,\alpha,\gamma)$$
$$\forall s,\nu > 0,\ \alpha \geq 0,\ \gamma \epsilon \mathbb{R}. \tag{0.10}$$

Here $\hat{\Psi}$ is normalized so that

$$\hat{\Psi}(s,\nu,\nu s,0) = 0 \qquad \forall s,\nu > 0 \tag{0.11}$$

and $\hat{\Psi}$ satisfies hypotheses which ensure that the integrals in (0.8) will be well behaved for a reasonable class of functions θ.

We assume that $\hat{\Psi}$ satisfies

$$\hat{\Psi},_1(s,\nu,\alpha,\gamma) + \nu\hat{\Psi},_3(s,\nu,\alpha,\gamma) \leq 0 \quad \forall s,\nu > 0,\ \alpha \geq 0,\ \gamma \epsilon \mathbb{R} \tag{0.12}$$

and thus by the main result obtained in [2] the constitutive relations (0.8) are compatible with thermodynamics and

$$\hat{\Psi},_j(s,\nu,\nu s,0) = 0 \qquad j = 1,2,3,4 \quad \forall s,\nu > 0. \tag{0.13}$$

Substitution of $(0.8)_3$ and (0.9) into the law of balance of energy (0.1) yields

$$\begin{aligned}
&\hat{c}_I(\theta(x,t),\ \bar{\theta}^t(x,\cdot),\ \bar{\theta}^t_x(x,\cdot))\theta_t(x,t) + \frac{\partial}{\partial x}\int_0^\infty \hat{Q}(s,\theta(x,t),\ \bar{\theta}^t(x,s),\ \bar{\theta}^t_x(x,s))ds \\
&\quad + \int_0^\infty \hat{E},_3(s,\theta(x,t),\ \bar{\theta}^t(x,s),\ \bar{\theta}^t_x(x,s))[\theta(x,t) - \theta(x,t-s)]ds \\
&\quad + \int_0^\infty \hat{E},_4(s,\theta(x,t),\ \bar{\theta}^t(x,s),\ \bar{\theta}^t_x(x,s))[\theta_x(x,t) - \theta_x(x,t-s)]ds = r(x,t) \\
&\qquad\qquad x \,\epsilon\, B,\ t \geq 0.
\end{aligned} \tag{0.14}$$

Here $\hat{Q}$ is given by

$$\hat{Q}(s,\nu,\alpha,\gamma) := -\nu\hat{\Psi},_4(s,\nu,\alpha,\gamma) \quad \forall\, s,\nu > 0,\ \alpha \geq 0,\ \gamma \epsilon \mathbb{R} \tag{0.15}$$

and

$$\hat{c}_I(\theta(s,t),\, \bar{\theta}^t(x,\cdot),\, \bar{\theta}^t_x(x,\cdot)) := \hat{e}'(\theta(x,t)) + \int_0^\infty \hat{E}_{,2}\,(s, \theta(x,t),\, \bar{\theta}^t(x,s)\,\bar{\theta}^t_x(x,s))ds \tag{0.16}$$

is the instantaneous heat capacity at $(\theta(x,t),\, \bar{\theta}^t(x,\cdot),\, \bar{\theta}^t_x(x,\cdot))$; the equilibrium heat capacity $\hat{c}_E(\nu)$ at the temperature ν is given by

$$\hat{c}_E(\nu) := \hat{e}'(\nu). \tag{0.17}$$

It is generally assumed in practice that the heat capacities are positive.

We seek a smooth solution to (0.14) subject to the initial conditions

$$\begin{aligned} \theta(x,t) &= \varphi(x,t) \qquad x \epsilon B,\, t < 0, \\ \theta(x,0) &= \theta_0(x) \qquad x \epsilon B, \end{aligned} \tag{0.18}$$

and appropriate boundary conditions if $B \neq \mathbb{R}$. Here $\varphi > 0$ and $\theta_0 > 0$ are prescribed smooth functions. Observe that (0.18) permits a temporal jump discontinuity in θ at $t = 0$. Even if such a discontinuity is present in the data one can obtain a solution that is smooth for $t \geq 0$ provided that θ_0 and $r(\cdot, 0)$ satisfy certain compatibility conditions at the endpoints of B.

It follows from the arguments of Gurtin and Pipkin [6] that compatibility with thermodynamics, strict positivity of the equilibrium heat capacity, and some assumptions of nondegeneracy imply that equation (0.14) is of hyperbolic type near equilibrium. The characteristic speeds for (0.14) are not constant and it is therefore possible that weak waves will be amplified and shocks will develop. On the other hand, equation (0.14) includes a natural damping mechanism induced by memory. It is not clear which effect is dominant. A great deal of insight into this question is given by Chen [3] who assumed the existence of solutions containing singularities called temperture rate waves, and obtained a formula for the amplitude of these waves. He found that an amplitude of small initial value decays as $t \to \infty$, and if the initial amplitude is large then blow up may occur in finite time. This suggests that when the data are close to equilibrium equation (0.14) has a global solution, whereas if the data are sufficiently far away from equilibrium the solution may develop singularities in finite time.

In order to keep the analysis relatively clean, while retaining the important features of (0.14) we treat the following special case in detail:

$$\begin{aligned}
\psi(x,t) &= \hat{\psi}(\theta(x,t)) - \frac{1}{\theta(x,t)}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds, \\
\eta(x,t) &= -\hat{\psi}'(\theta(x,t)) - \frac{1}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds, \\
q(x,t) &= \int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))ds.
\end{aligned} \tag{0.19}$$

Here $\hat{\psi} : (0,\infty) \to \mathbb{R}$, $a : [0,\infty) \to \mathbb{R}$, and $F : \mathbb{R} \to \mathbb{R}$ are smooth functions with $a \epsilon W^{3,1}(0,\infty)$ and $F(0) = 0$. We assume that

$$a \text{ is convex}, \quad F(\gamma) \geq 0 \qquad \forall \gamma \epsilon \mathbb{R}; \tag{0.20}$$

the arguments used in [2] can be applied in the present setting to show that (0.20) implies that the constitutive equations (0.19) are compatible with thermodynamics. We note that by (0.20) we have

$$a' \leq 0,\ a \geq 0,\ F'(0) = 0,\ F''(0) \geq 0. \tag{0.21}$$

The corresponding equation for e is

$$e(x,t) = \hat{e}(\theta(x,t)) - \frac{2}{\theta(x,t)}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds, \tag{0.22}$$

where $\hat{e}$ is as in $(0.10)_1$. Thus (0.1) yields

$$\begin{aligned}
&(\hat{e}'(\theta(x,t)) + \frac{2}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds)\theta_t(x,t) \\
&\quad + \int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))\bar{\theta}_{xx}^t(x,s)ds \\
&\quad - \frac{2}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))[\theta_x(x,t)-\theta_x(x,t-s)]ds = r(x,t) \\
&\qquad\qquad x \in B,\ t \geq 0.
\end{aligned} \tag{0.23}$$

We establish global existence and asymptotic stability of smooth solutions to the initial-value problem (0.23), (0.18) for smooth data (r, φ, θ_0) that are close to equilibrium. We treat Dirichlet and Neumann boundary conditions as well as the problem with $B = \mathbb{R}$. We also make some remarks concerning the extension of our work to the initial-value problem (0.14), (0.18).

To indicate the nature of our results let us consider the case where $B = [0,1]$, $\varphi \equiv \theta_0 \equiv \theta^*$, with Dirichlet boundary conditions

$$\theta(0,t) = \theta(1,t) = \theta^* \qquad t \geq 0, \tag{0.24}$$

where $\theta^* > 0$ is a given constant.

In order to prove global existence of solutions to (0.23), (0.24), (0.18) we need to make additional assumptions on the constitutive relations and on the data. Concerning the constitutive equations we require that

$$a \not\equiv 0 \tag{0.25}$$

and we strengthen the inequality $(0.21)_4$ to the strict inequality

$$F''(0) > 0. \tag{0.26}$$

These two conditions imply that the linearized relation for the heat flux is nontrivial. We also assume that the equilibrium heat capacity is strictly positive, i.e.[2]

$$\hat{e}'(\nu) > 0 \qquad \forall \nu > 0. \tag{0.27}$$

Assumptions (0.25) - (0.27) imply that equation (0.23) is hyperbolic near equilibrium. Since we have dependence on the summed history of the temperature gradient (for which we do not obtain a pointwise bound), we need to make a growth restriction on F that is related to the decay rate of a. In addition, we assume that the heat supply r is smooth, decays with time, and is small in a sense that will be stated more precisely later. Moreover, to ensure the existence of smooth classical solutions the heat supply r must satisfy the condition

$$r(0,0) = r(1,0) = r_t(0,0) = r_t(1,0) = 0 \ , \tag{0.28}$$

which guarantees compatibility of the data with the boundary conditions at $t = 0$. We note that our assumptions imply that a is a strongly positive definite kernel in the sense of [11]. Inequalities for such kernels play an essential role in the proof of global existence.

Observe that if $r \equiv 0$, then $\theta \equiv \theta^*$ is a solution. We look for classical solutions to (0.23), (0.24), (0.18) near the prescribed equilibrium temperature θ^* for $t \geq 0$. We show that (0.23), (0.24), (0.18) has a unique solution $\theta > 0$ with θ, θ_x, θ_t, θ_{xx}, θ_{xt}, θ_{tt}, θ_{xxx}, θ_{xxt}, $\theta_{xtt}, \theta_{ttt} \in C([0,\infty); L^2(0,1))$ and θ, θ_x, θ_t, θ_{xx}, θ_{xt}, $\theta_{tt} \in L^2((0,\infty); L^2(0,1)) \cap L^\infty((0,\infty); L^2(0,1))$. Moreover, as $t \to \infty$, $\theta(\cdot,t) \to \theta^*$ and $\theta_x(\cdot,t)$, $\theta_t(\cdot,t) \to 0$ uniformly

[2] For our purposes it suffices to assume that $\hat{e}'(\theta^*) > 0$; however, assumption (0.27) is in accord with experience and leads to certain simplifications in the proofs of Theorems 1.2 and 1.3.

on $[0,1]$. An analogous result can be obtained for Neumann boundary conditions as well as for the problem with $B = \mathbb{R}$.

The arguments used to prove global existence in [5], [10], and [12] for MacCamy's model are similar in spirit to the arguments used here. The primary differences between our existence proof and those for MacCamy's model arise from the dependence of e on the summed history of θ_x. This dependence complicates the analysis and necessitates the use of a new inequality for strongly positive definite kernels. Global existence is obtained by deriving a priori estimates; in these derivations we exploit the compatibility of our constitutive relations with thermodynamics, i.e. we make use of the entropy inequality (0.2). It is interesting to note that one can obtain an existence result for (0.23), (0.24), (0.18) without utilizing the thermodynamical restrictions, provided the linearized equation has the appropriate features. However, the compatibility conditions imposed on our constitutive relations by the thermodynamical restrictions allow us to establish a global existence result under less restrictive assumptions on the data.

The paper is organized as follows. Precise statements of global existence results are given in Section 1. Section 2 is concerned with appropriate local existence results and with properties of strongly positive definite kernels relevant to our needs. Section 3 is devoted to the proof of the theorems stated in Section 1; the proof for the problem with Dirichlet boundary conditions is discussed in detail and remarks are made concerning other boundary conditions.

1. Statement of Results

We first consider the problem

$$
\begin{aligned}
(\hat{e}'(\theta(x,t)) + \frac{2}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds)\theta_t(x,t) \\
+ \int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))\bar{\theta}_{xx}^t(x,s)ds \\
-\frac{2}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))[\theta_x(x,t)-\theta_x(x,t-s)]ds = r(x,t)
\end{aligned}
\tag{1.1}
$$

$$x \,\epsilon\, [0,1],\ t \geq 0\,,$$

$$\theta(x,t) = \theta^* \qquad x \,\epsilon\, [0,1], t < 0\,, \tag{1.2}$$

$$\theta(x,0) = \theta_0(x) \qquad x \,\epsilon\, [0,1]\,, \tag{1.3}$$

$$\theta(0,t) = \theta(1,t) = \theta^* \qquad t \geq 0\,. \tag{1.4}$$

Here, $\theta^* > 0$ is a given constant and $\theta_0 : [0,1] \to (0,\infty)$ is a prescribed smooth function.

Concerning $\hat{e}$, F, and a we require

$$\hat{e} \,\epsilon\, C^4(0,\infty), \tag{1.5}$$

$$\hat{e}'(\nu) > 0 \qquad \forall \nu > 0; \tag{1.6}$$

$$F \,\epsilon\, C^5(\mathbb{R}), \tag{1.7}$$

$$F(0) = 0, \quad F''(0) > 0, \quad F(\gamma) \geq 0 \quad \forall \gamma \,\epsilon\, \mathbb{R}, \tag{1.8}$$

and there are constants $K > 0$, $k > 1$ such that

$$|F^{(j)}(\xi) - F^{(j)}(0)| \leq K(|\xi| + |\xi|^k) \quad j = 0,1,2,3,4,5,\ \xi \epsilon \mathbb{R}; \tag{1.9}$$

$$a \epsilon W^{3,1}(0,\infty), \quad a \ \text{ is strongly positive definite,} \quad a'' \geq 0, \tag{1.10}$$

and

$$\int_0^\infty |a'(z)|z^k dz,\ \int_0^\infty \int_s^\infty |a'(z)|z^k dz ds,\ \int_0^\infty |a''(z)|z^k dz,$$
$$\int_0^\infty \int_s^\infty |a''(z)|z^k dz ds,\ \int_0^\infty |a'''(z)|z^k dz < \infty. \tag{1.11}$$

The definition of a strongly positive definite kernel is given in the next section. For now, it suffices to know that

(i) $(1.10)_{1,2}$ implies $a(0) > 0$;

(ii) if $a \in W^{3,1}(0,\infty)$, $a \not\equiv 0$, and $a'' \geq 0$ then a is strongly positive definite.

The data are assumed to have the following regularity:

$$\theta_0 \in H^3(0,1), \tag{1.12}$$

$$r, r_x, r_t, r_{xt}, r_{tt} \in C([0,\infty); L^2(0,1)) \cap L^2((0,\infty); L^2(0,1)) \cap L^\infty((0,\infty); L^2(0,1)), \tag{1.13}$$

$$r(\cdot,0) \in H^2(0,1),\ r_{ttt} \in L^2((0,\infty); L^2(0,1)). \tag{1.14}$$

We also assume that the following compatibility conditions hold on the boundary:

$$\theta_0(0) = \theta_0(1) = \theta^*, \tag{1.15}$$

$$r(0,0) = r(1,0) = 0, \tag{1.16}$$

$$r_t(0,0) = a(0)F''(0)\,(-\theta_0''(0) + \frac{2}{\theta^*}\theta_0'(0)^2), \tag{1.17}$$

$$r_t(1,0) = a(0)F''(0)\,(-\theta_0''(1) + \frac{2}{\theta^*}\theta_0'(1)^2). \tag{1.18}$$

The interpretation of (1.15) is clear; conditions (1.16) - (1.18) ensure that $\theta_t(\cdot,0)$ and $\theta_{tt}(\cdot,0)$ vanish on the boundary. In order to state our results, it is convenient to define

$$\Theta_0 := \int_0^1 ([\theta_0(x) - \theta^*]^2 + \theta_0'(x)^2 + \theta_0''(x)^2)dx \tag{1.19}$$

and

$$\mathbf{R}_0 := \sup_{t\geq 0}\int_0^1 (r^2+r_t^2)(x,t)dx + \int_0^1 r_x^2(x,0)dx + (\sup_{\substack{x\epsilon[0,1]\\ t\geq 0}} |r(x,t)|)^2$$
$$+\int_0^\infty\int_0^1 (r^2+r_t^2+r_{tt}^2)(x,t)dxdt. \tag{1.20}$$

We establish the following result.

Theorem 1.1: *Assume that* (1.5) - (1.11) *hold. Then there is a constant* $\delta > 0$ *such that for all* θ_0 *and* r *satisfying* (1.12) - (1.18) *and*

$$\Theta_0 + R_0 \leq \delta^2, \tag{1.21}$$

the initial-value problem (1.1), (1.2), (1.3), (1.4) *has a unique solution* $\theta > 0$ *with*

$$\theta,\theta_x,\theta_t,\theta_{xx},\theta_{xt},\theta_{tt},\theta_{xxx},\theta_{xxt},\theta_{xtt},\theta_{ttt}\,\epsilon\, C([0,\infty);L^2(0,1)) \tag{1.22}$$

and

$$\theta,\theta_x,\theta_t,\theta_{xx},\theta_{xt},\theta_{tt}\,\epsilon\, L^\infty((0,\infty);L^2(0,1))\cap L^2((0,\infty);L^2(0,1)). \tag{1.23}$$

Moreover, as $t\to\infty$

$$\theta(\cdot,t)\to\theta^* \tag{1.24}$$

and

$$\theta_x(\cdot,t),\ \theta_t(\cdot,t)\to 0 \ \ \textit{uniformly on}\ \ [0,1]. \tag{1.25}$$

Remark 1.1: *The constant* δ *in Theorem 1.1 depends on* θ^* *and on properties of the functions appearing in the constitutive relations.*

Remark 1.2: *By the Sobolev embedding theorem,* (1.22) *implies that* $\theta\,\epsilon\, C^2([0,1]\times[0,\infty))$.

A result analogous to Theorem 1.1 can be established if we replace the Dirichlet boundary conditions (1.4) with Neumann boundary conditions

$$\theta_x(0,t) = \theta_x(1,t) = 0 \qquad t \geq 0. \tag{1.26}$$

Remark 1.3: *Under the assumptions of Theorem 1.2 below* (1.26) *holds if and only if*

$$q(0,t) = q(1,t) = 0 \quad t \geq 0. \tag{1.27}$$

(*Recall that the heat flux, q, is given by* $(0.19)_3$.) *It is obvious that* (1.26) *implies* (1.27). *In order to show that* (1.27) *implies* (1.26) *we first differentiate the relation for the heat flux* $(0.19)_3$ *with respect to t on the boundary, making use of* (1.2). *We then add and subtract terms to obtain the identity*

$$\begin{aligned} & a(0)\theta_x(\xi,t) + \int_0^t a'(t-s)\theta_x(\xi,s)ds \\ & \qquad = \frac{1}{F''(0)}\frac{\partial}{\partial t}\int_0^t \theta_x(\xi,s)\int_{t-s}^{\infty} a'(y)\int_0^1 (F''(z\bar{\theta}_x^t(\xi,y)) - F''(0))dzdyds \\ & \qquad \xi = 0,1, \qquad t \geq 0. \end{aligned} \tag{1.28}$$

We can now solve (1.28) *for θ_x and make use of Lemma 2.3 below to show that* (1.26) *is the only continuous solution of* (1.28) *that vanishes at $t = 0$.*

We now require that r satisfy (1.13), (1.14), and

$$r \in L^1((0,\infty); L^2(0,1)); \tag{1.29}$$

in addition we assume that the compatibility conditions

$$\theta_0'(0) = \theta_0'(1) = 0, \tag{1.30}$$

$$r(0,0) = r(1,0) = 0, \tag{1.31}$$

$$r_t(0,0) = -a(0)F''(0)\theta_0''(0), \tag{1.32}$$

$$r_t(1,0) = -a(0)F''(0)\theta_0''(1) \tag{1.33}$$

hold.

Theorem 1.2: *Assume that* (1.5) - (1.11) *hold. Then there is a constant* $\delta > 0$ *such that for every* θ_0 *and* r *that satisfy* (1.12) - (1.14), (1.29) - (1.33) *and*

$$\Theta_0 + R_0 + \left(\int_0^\infty \left(\int_0^1 r(x,t)^2 dx\right)^{\frac{1}{2}} dt\right)^2 \leq \delta^2 \tag{1.34}$$

the initial-value problem (1.1), (1.2), (1.3), (1.26) *has a unique solution* $\theta > 0$ *with*

$$\theta, \theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt}, \theta_{xxx}, \theta_{xxt}, \theta_{xtt}, \theta_{ttt} \in C([0,\infty); L^2(0,1)), \tag{1.35}$$

$$\theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt} \in L^\infty((0,\infty); L^2(0,1)) \cap L^2((0,\infty); L^2(0,1)), \tag{1.36}$$

and

$$\theta \in L^\infty((0,\infty);\, L^2(0,1)). \tag{1.37}$$

Furthermore, as $t \to \infty$, $\theta(\cdot,t)$ *converges to a constant* $\theta^{**} > 0$ *uniformly on* $[0,1]$ *and*

$$\theta_x(\cdot,t),\ \theta_t(\cdot,t) \to 0 \quad \textit{uniformly on} \quad [0,1]. \tag{1.38}$$

Remark 1.4: *The value of* θ^{**} *can be determined from equation* (1.1) *as follows. If the assumptions of Theorem* 1.2 *hold and* θ *is a solution of* (1.1), (1.2), (1.3), (1.26) *then integrating* (1.1) *over* $[0,1] \times [0,t]$, $t > 0$, *and passing to the limit as* $t \to \infty$ *yields*

$$\hat{e}(\theta^{**}) = \int_0^1 \hat{e}(\theta_0(x))dx + \int_0^\infty \int_0^1 r(x,t)dxdt. \tag{1.39}$$

By (1.6) $\hat{e}$ *is strictly monotone and hence there is a unique solution* θ^{**} *of* (1.39).

Let us now consider the problem stated below in which the heat conductor occupies the entire real line:

$$\begin{aligned}(\hat{e}'(\theta(x,t)) + \frac{2}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds)\,\theta_t(x,t)\\ + \int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))\bar{\theta}_{xx}^t(x,s)ds\\ - \frac{2}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))[\theta_x(x,t)-\theta_x(x,t-s)]ds = r(x,t)\\ x \in \mathbb{R},\ t \geq 0\,,\end{aligned} \tag{1.40}$$

$$\theta(x,t) = \theta^* \qquad x \in \mathbb{R}, t < 0\,, \tag{1.41}$$

$$\theta(x,0) = \theta_0(x) \qquad x \in \mathbb{R}\,. \tag{1.42}$$

We assume

$$\theta_0 - \theta^* \in H^3(\mathbb{R}), \tag{1.43}$$

$$r_x, r_t, r_{xt}, r_{tt} \in C([0,\infty); L^2(\mathbb{R})) \cap L^2((0,\infty); L^2(\mathbb{R})) \cap L^\infty((0,\infty); L^2(\mathbb{R})), \tag{1.44}$$

$$r \in C([0,\infty); L^2(\mathbb{R})) \cap L^1((0,\infty); L^2(\mathbb{R})) \cap L^\infty((0,\infty); L^2(\mathbb{R})), \tag{1.45}$$

$$r(\cdot,0) \in H^2(\mathbb{R}),\ r_{ttt} \in L^2((0,\infty); L^2(\mathbb{R})). \tag{1.46}$$

Note that (1.45) implies $r \in L^2((0,\infty); L^2(\mathbb{R}))$. We define

$$\Theta_1 := \int_{-\infty}^\infty ([\theta_0(x) - \theta^*]^2 + \theta_0'(x)^2 + \theta_0''(x)^2)dx \tag{1.47}$$

and

$$\begin{aligned}R_1 := \sup_{t\geq 0}\int_{-\infty}^\infty (r^2 + r_t^2)(x,t)dx + \int_{-\infty}^\infty r_x^2(x,0)dx + (\sup_{\substack{x\in\mathbb{R}\\ t\geq 0}} |r(x,t)|)^2\\ + \int_0^\infty\int_{-\infty}^\infty (r^2 + r_t^2 + r_{tt}^2)(x,t)dxdt + (\int_0^\infty(\int_{-\infty}^\infty r(x,t)^2dx)^{\frac{1}{2}}dt)^2.\end{aligned} \tag{1.48}$$

Theorem 1.3: *If* (1.5) - (1.11) *hold, then there is a constant* $\delta > 0$ *such that when* θ_0 *and* r *satisfy* (1.43) - (1.46) *and*

$$\Theta_1 + R_1 \leq \delta^2, \tag{1.49}$$

the initial-value problem (1.40), (1.41), (1.42) *has a unique solution* $\theta > 0$ *with*

$$\theta - \theta^*, \theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt}, \theta_{xxx}, \theta_{xxt}, \theta_{xtt}, \theta_{ttt} \in C([0,\infty); L^2(\mathbb{R})), \tag{1.50}$$

$$\theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt} \in L^\infty((0,\infty); L^2(\mathbb{R})) \cap L^2((0,\infty); L^2(\mathbb{R})), \tag{1.51}$$

and

$$\theta - \theta^* \in L^\infty((0,\infty); L^2(\mathbb{R})). \tag{1.52}$$

In addition, as $t \to \infty$,

$$\theta(\cdot,t) \to \theta^* \quad \textit{uniformly on} \quad \mathbb{R} \tag{1.53}$$

and

$$\theta_x(\cdot,t),\ \theta_t(\cdot,t) \to 0 \quad \textit{uniformly on} \quad \mathbb{R} \quad \textit{and in} \quad L^2(\mathbb{R}). \tag{1.54}$$

Remark 1.5: *A detailed proof of Theorem 1.1 is given in Section 3. With some minor modifications the argument used to establish Theorem 1.1 can be applied to prove Theorems 1.2 and 1.3; these modifications are discussed in Section 3.*

Remark 1.6: *Assumption (1.11) is not the weakest possible to obtain the global estimates of Section 3. However, in order to establish local existence the replacement of (1.11) with a weaker assumption would necessitate a much more sophisticated argument than the one used in Chapter III of [1].*

The results established here can be modified and extended, as is illustrated below.

(i) Weak Solutions: Using a density argument one can show that under weaker assumptions on the data, our initial-value problems have a unique, globally defined, weak solution. More precisely, for instance in Theorem 1.1, if we replace (1.12) - (1.18) with

$$\theta_0 \in H^2(0,1), \tag{1.55}$$

$$r, r_t \in C([0,\infty); L^2(0,1)) \cap L^2((0,\infty); L^2(0,1)) \cap L^\infty((0,\infty); L^2(0,1)), \tag{1.56}$$

$$r \in L^\infty((0,1)\times(0,\infty)),\ r(\cdot,0) \in H^1(0,1),\ r_{tt} \in L^2((0,\infty); L^2(0,1)), \tag{1.57}$$

and

$$\theta_0(0) = \theta_0(1) = \theta^*, \quad r(0,0) = r(1,0) = 0, \tag{1.58}$$

then the result of Theorem 1.1 is true with (1.22) replaced with

$$\theta, \theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt} \in C([0,\infty); L^2(0,1)). \tag{1.59}$$

(ii) Nonequilibrium History: Results analogous to Theorems 1.1, 1.2, and 1.3 can be obtained if a more general history is prescribed. For example, a result similar to Theorem 1.1 can be established if (1.2) is replaced by

$$\theta(x,t) = \varphi(x,t) \qquad x \in [0,1], t < 0, \tag{1.60}$$

where $\varphi : [0,1]\times(-\infty,0] \to (0,\infty)$ satisfies

$$\begin{aligned} &\varphi_x, \varphi_{xx}, \varphi_{xt}, \varphi_{xxx}, \varphi_{xxt}, \varphi_{xtt} \in C((-\infty,0]; L^2(0,1)) \\ &\cap L^2((-\infty,0); L^2(0,1)) \cap L^\infty((-\infty,0); L^2(0,1)), \end{aligned} \tag{1.61}$$

$$\int_0^\infty a'(s)F'(\bar{\varphi}_x^0(\cdot,s))ds \in H^3(0,1), \tag{1.62}$$

and the compatibility conditions (1.16) - (1.18) are modified accordingly. In addition, the quantity

$$\begin{aligned}\Phi := \sup_{t\epsilon(-\infty,0)} \int_0^1 (\varphi_x^2 + \varphi_{xx}^2 + \varphi_{xt}^2)(x,t)dx \\ + \int_{-\infty}^0 \int_0^1 (\varphi_x^2 + \varphi_{xx}^2 + \varphi_{xt}^2)(x,t)dxdt \\ + \int_0^1 \left(\int_0^\infty a'(s)\frac{\partial^2}{\partial x^2}F'(\bar{\varphi}_x^0(x,s))ds\right)^2 dx\end{aligned} \tag{1.63}$$

must be sufficiently small, i.e. condition (1.21) is to be replaced with

$$\Theta_0 + \Phi + R_0 \le \delta^2. \tag{1.64}$$

In Section 3 we discuss modifications needed in order to adapt the proof of Theorem 1.1 to this case. We note that for the analogue of Theorem 1.2, if we assume that

$$\varphi_x(0,t) = \varphi_x(1,t) = 0 \qquad t \le 0 \tag{1.65}$$

then, following the procedure discussed in Remark 1.3, we can show that (1.26) is equivalent to (1.27).

(iii) General Integral Models: These results can be extended to the case when the constitutive equations (0.8) are considered. In these equations the dependence on the summed history of θ is nontrivial; hence a term involving $\theta(x,t)$ appears in the analogue of (1.1). In the corresponding linearized equation the coefficient of $\theta(x,t)$ is

$$E^* := \int_0^\infty \hat{E}_{,3}(s,\theta^*,\theta^* s,0)ds; \tag{1.66}$$

one can show that compatibility with thermodynamics implies that E^* is nonnegative and hence the methods we use here can be adopted to produce analogous results to those stated in Theorems 1.1, 1.2, and 1.3. The precise statement of the technical assumptions required would be very complicated and not very illuminating, e.g. the mapping

$$s \mapsto Q_{,4}(s, \theta^*, \theta^* s, 0) \tag{1.67}$$

would have to be such that our assumptions on

$$s \mapsto a'(s) F''(0) \tag{1.68}$$

would hold. We will not discuss this case in further detail.

2. Preliminaries

We begin by stating a local existence result for (1.1), (1.2), (1.3), (1.4). We first note that equation (1.1) is hyperbolic near equilibrium, but may *lose its evolutionary character* at states sufficiently far from equilibrium. In order to ensure that (1.1), (1.2), (1.3), (1.4) is well posed we assume that θ_0 is close to equilibrium in the sense described below. We choose $\varepsilon \in (0, \theta^*)$ sufficiently small so that there are constants $e^*, q^* > 0$ with the following property:

$$\hat{e}'(w(x,t)) + \frac{2}{w(x,t)^2} \int_0^\infty a'(s) F(\bar{w}_x^t(x,s))ds \geq e^* \quad \forall x \in [0,1], t \in [0,T], \tag{2.1}$$

and

$$-\int_0^\infty a'(s) F''(\bar{w}_x^t(x,s))ds \geq q^* \quad \forall x \in [0,1], t \in [0,T], \tag{2.2}$$

for every $T > 0$ and every $w \in L^\infty((-\infty,T); H^1(0,1))$ satisfying

$$|w(x,t) - \theta^*|, |w_x(x,t)| \leq \varepsilon \quad \forall x \in [0,1], t \in (-\infty, T]. \tag{2.3}$$

Such a choice is possible by virtue of our assumptions on a and F. (Indeed, the left-hand sides of (2.1) and (2.2) are strictly positive when $w(x,t) \equiv \theta^*$. A simple perturbation about $w = \theta^*$ guarantees the existence of a suitable ε. In fact, we may take $e^* = \frac{1}{2}\hat{e}'(\theta^*)$ and $q^* = \frac{1}{2}a(0)F''(0)$.) We assume that θ_0 satisfies

$$|\theta_0(x) - \theta^*| \, , \; |\theta_0'(x)| \leq \eta \quad \forall x \in [0,1], \tag{2.4}$$

for some $\eta \in (0, \varepsilon)$.

We can now state the following lemma.

Lemma 2.1: *Assume that* (1.5) - (1.18) *and* (2.4) *are satisfied. Then the initial-value problem* (1.1), (1.2), (1.3), (1.4) *has a unique solution* $\theta > 0$, *defined on a maximal time interval* $[0, T_0)$, $T_0 > 0$, *with*

$$\theta, \theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt}, \theta_{xxx}, \theta_{xxt}, \theta_{xtt}, \theta_{ttt} \in C([0, T_0); L^2(0,1)) \tag{2.5}$$

and

$$|\theta(x,t) - \theta^*|, |\theta_x(x,t)| < \varepsilon \quad \forall x \in [0,1], t \in [0, T_0). \tag{2.6}$$

Moreover, if

$$\sup_{\substack{x \in [0,1] \\ t \in [0,T_0)}} |\theta(x,t) - \theta^*|, \sup_{\substack{x \in [0,1] \\ t \in [0,T_0)}} |\theta_x(x,t)| < \varepsilon \tag{2.7}$$

and

$$\sup_{t \in [0,T_0)} \int_0^1 (\theta^2 + \theta_x^2 + \theta_t^2 + \theta_{xx}^2 + \theta_{xt}^2 + \theta_{tt}^2 + \theta_{xxx}^2 + \theta_{xxt}^2 + \theta_{xtt}^2 + \theta_{ttt}^2)(x,t)dx < \infty \tag{2.8}$$

then $T_0 = \infty$.

A result analogous to Lemma 2.1 can be established if we replace (1.4) by (1.26) (i.e. if instead of Dirichlet boundary conditions we consider Neumann boundary conditions) and (1.15) - (1.18) with (1.29) - (1.33). Similarly to (1.1), (1.2), (1.3), (1.4), the initial-value problem (1.1), (1.2), (1.3), (1.26) has a unique solution θ defined on a maximal time interval $[0, T_0), T_0 > 0$ satisfying (2.5) and (2.6). One can also obtain a corresponding result for the case when the heat conductor occupies the entire real line; the assumptions required in this case would be the analogues on $\mathbb{R}$ of the assumptions stated above.

The proof of Lemma 2.1 is given in Chapter III of [1][3] so we omit *the details. It is* interesting to note that although compatibility of the constitutive relations (0.19), (0.22) with thermodynamics determines the form of equation (1.1) it plays no further role in the proof of Lemma 2.1. However, a bootstrapping argument, in which the thermodynamical restrictions play an essential part, can be applied to strengthen the result described in Lemma 2.1. More precisely, one can show that under the assumptions of Lemma 2.1, if θ satisfies (1.1), (1.2), (1.3), (1.4) on a maximal time interval $[0, T_0), T_0 > 0$ (and hence θ satisfies the entropy inequality (0.2)), then a bound on the $L^\infty([0, T_0); L^2(0,1))$ norms of θ and its derivatives through order two implies that there is a bound on the aforementioned norms of third order derivatives of θ. Hence, one can establish the following lemma.

Lemma 2.2: *Suppose that the assumptions of Lemma* 2.1 *hold and that* θ *is a solution of* (1.1), (1.2), (1.3), (1.4) *on a maximal time interval* $[0, T_0), T_0 > 0$. *If* θ *satisfies* (2.7) *and*

$$\sup_{t \in [0,T_0)} \int_0^1 (\theta^2 + \theta_x^2 + \theta_t^2 + \theta_{xx}^2 + \theta_{xt}^2 + \theta_{tt}^2)(x,t)dx < \infty, \tag{2.9}$$

then $T_0 = \infty$.

[3] Assumption $(1.15)_2$ of Chapter III of [1] does not suffice to ensure that $\theta_t(\cdot, 0^+) \in H^2(0,1)$. One needs to make the additional assumption that

$$\int_0^\infty a'(s)F'(\bar{\varphi}_x^0(\cdot, s))ds \in H^3(0,1),$$

where $\varphi : [0,1] \times (-\infty, 0] \to (0, \infty)$ is a prescribed general history, i.e.

$$\theta(x,t) = \varphi(x,t) \quad x \in [0,1], t < 0.$$

However, the arguments used to prove Theorem 1.1 of Chapter III of [1] remain valid.

Remark 2.1: *If θ is a solution of* (1.1), (1.2), (1.3), (1.4) *then θ satisfies the entropy inequality* (0.2), *where the entropy and the heat flux are given by* $(0.19)_2$ *and* $(0.19)_3$, *i.e.*

$$-\frac{\partial}{\partial x}\Big(\frac{1}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))ds\Big) \leq \frac{\partial}{\partial t}(-\hat{\psi}'(\theta(x,t)) \\ -\frac{1}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds) - \frac{r(x,t)}{\theta(x,t)}. \tag{2.10}$$

Recall that

$$\hat{e}(\nu) := \hat{\psi}(\nu) - \nu\hat{\psi}'(\nu) \quad \forall \nu > 0, \tag{2.11}$$

hence (1.5) *implies*

$$\hat{\psi}'' \in C(0,\infty). \tag{2.12}$$

Before proving Lemma 2.2 we introduce the following definition. For $T > 0$ and $0 < h < T$, we define the forward difference operator Δ_h (with respect to the time variable) by

$$(\Delta_h w)(x,t) := w(x,t+h) - w(x,t) \quad \forall x \in [0,1], t \in [0,T-h] \tag{2.13}$$

for every $w \in C([0,T];L^2(0,1))$.

Proof of Lemma 2.2: Let θ be a solution of (1.1), (1.2), (1.3), (1.4) on a maximal time interval $[0,T_0), T_0 > 0$, such that (2.7) holds. Our aim is to show that if $T_0 < \infty$ then

$$\sup_{t\in[0,T_0)}\int_0^1 (\theta^2 + \theta_x^2 + \theta_t^2 + \theta_{xx}^2 + \theta_{xt}^2 + \theta_{tt}^2)(x,t)dx = \infty. \tag{2.14}$$

For this purpose it is convenient to introduce the quantities

$$\gamma_2(t) := \sup_{s\in[0,t]}\int_0^1 (\theta^2 + \theta_x^2 + \theta_t^2 + \theta_{xx}^2 + \theta_{xt}^2 + \theta_{tt}^2)(x,s)dx \quad t \in [0,T_0), \tag{2.15}$$

$$\gamma_3(t) := \sup_{s\epsilon[0,t]} \int_0^1 (\theta^2 + \theta_x^2 + \theta_t^2 + \theta_{xx}^2 + \theta_{xt}^2 + \theta_{tt}^2 + \theta_{xxx}^2 + \theta_{xxt}^2 + \theta_{xtt}^2 + \theta_{ttt}^2)(x,s)dx \quad t\,\epsilon\,[0,T_0), \tag{2.16}$$

$$\Theta := \int_0^1 (\theta_0(x)^2 + \theta_0'(x)^2 + \theta_0''(x)^2 + \theta_0'''(x)^2)dx, \tag{2.17}$$

and

$$R := \sup_{t\epsilon(0,\infty)} \int_0^1 (r^2 + r_x^2 + r_t^2 + r_{xt}^2 + r_{tt}^2)(x,t)dx + \int_0^1 r_{xx}^2(x,0)dx + \int_0^\infty \int_0^1 (r^2 + r_x^2 + r_t^2 + r_{xt}^2 + r_{tt}^2 + r_{ttt}^2)(x,t)dxdt. \tag{2.18}$$

In the following calculations we make use of the inequalities

$$(\sum_{i=1}^N A_i)^2 \leq N \sum_{i=1}^N A_i^2 \qquad A_1, ..., A_N \,\epsilon\, \mathbb{R}, \tag{2.19}$$

$$|AB| \leq \frac{A^2}{4\lambda} + \lambda B^2 \quad A, B\,\epsilon\,\mathbb{R}, \lambda > 0, \tag{2.20}$$

and

$$\|A * B\|_{L^p((0,T);L^2(0,1))} \leq \|A\|_{L^1(0,\infty)} \|B\|_{L^p((0,T);L^2(0,1))} \tag{2.21}$$

for every $T > 0, A\,\epsilon\,L^1(0,\infty)$, and $B\,\epsilon\,L^p((0,T);L^2(0,1))$, where $1 \leq p \leq \infty$ and $A * B$ denotes the convolution of A with B. We use Γ to denote a (possible large) positive generic constant which is independent of θ_0, r, and T_0.

We first differentiate equation (1.1) twice with respect to t and then apply the forward difference operator Δ_h to the resulting expression. We multiply the new equation by $\Delta_h \theta_{tt}$ and integrate over $[0,1] \times [0,t], t\,\epsilon\,(0,T_0)$. After several integrations by parts, we divide both sides by h^2 and let $h \downarrow 0$ to obtain the identity

$$
\begin{aligned}
&\frac{1}{2}\int_0^1(\hat{e}'(\theta(x,t))+\frac{2}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds)\theta_{ttt}^2(x,t)dx \\
&\quad -\frac{1}{2}\int_0^1\int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))ds\theta_{xtt}^2(x,t)dx \\
&\quad =-\int_0^t\int_0^1\frac{\partial}{\partial x}(\frac{1}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz)\theta_{ttt}^2(x,s)dxds \\
&\quad +\frac{1}{2}\int_0^1\hat{e}'(\theta_0(x))\theta_{ttt}^2(x,0)dx+\frac{1}{2}a(0)F''(0)\int_0^1\theta_{xtt}^2(x,0)dx \\
&\quad +\frac{1}{2}\int_0^t\int_0^1\frac{\partial}{\partial s}(\hat{e}'(\theta(x,s))+\frac{2}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz)\theta_{ttt}^2(x,s)dxds \\
&\quad -\frac{1}{2}\int_0^t\int_0^1\frac{\partial}{\partial s}\int_0^\infty a'(z)F''(\bar{\theta}_x^s(x,z))dz\,\theta_{xtt}^2(x,s)dxds \\
&\quad +\int_0^t\int_0^1\frac{\partial^3}{\partial s^3}(r(x,s) \\
&\qquad -\frac{2}{\theta(x,s)}\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dz)\theta_{ttt}(x,s)dxds \\
&\quad -\int_0^t\int_0^1\theta_t(x,s)\frac{\partial^3}{\partial s^3}(\hat{e}'(\theta(x,s)) \\
&\qquad +\frac{2}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz)\theta_{ttt}(x,s)dxds \\
&\quad -3\int_0^t\int_0^1\frac{\partial}{\partial s}(\theta_{tt}(x,s)\frac{\partial}{\partial s}(\hat{e}'(\theta(x,s)) \\
&\qquad +\frac{2}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz))\theta_{ttt}(x,s)dxds \\
&\quad +\int_0^t\int_0^1\frac{\partial^2}{\partial s^2}\int_0^s a'(z)F''(\bar{\theta}_x^s(x,z))\theta_{xx}(x,s-z)dz\theta_{ttt}(x,s)dxds \\
&\quad +\int_0^t\int_0^1\frac{\partial^2}{\partial s^2}\int_0^s a'(z)F'''(\bar{\theta}_x^s(x,z))\bar{\theta}_{xx}^s(x,z)\theta_x(x,s-z)dz\theta_{ttt}(x,s)dxds \\
&\quad -\int_0^t\int_0^1\theta_x(x,s)\frac{\partial^2}{\partial s^2}\int_0^s a'(z)F'''(\bar{\theta}_x^s(x,z))\bar{\theta}_{xx}^s(x,z)dz\theta_{ttt}(x,s)dxds \\
&\quad -2\int_0^t\int_0^1\theta_{xt}(x,s)\frac{\partial}{\partial s}\int_0^s a'(z)F'''(\bar{\theta}_x^s(x,z))\bar{\theta}_{xx}^s(x,z)dz\theta_{ttt}(x,s)dxds \\
&\quad +\int_0^t\int_0^1\theta_x(x,s)\frac{\partial^3}{\partial s^3}(\frac{2}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz)\theta_{ttt}(x,s)dxds \\
&\quad +3\int_0^t\int_0^1\frac{\partial}{\partial s}(\theta_{xt}(x,s)\frac{\partial}{\partial s}(\frac{2}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz))\theta_{ttt}(x,s)dxds
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
&-\int_0^t\int_0^1 \frac{\partial^2}{\partial s^2}\int_0^\infty a'(z)F''(\bar\theta_x^s(x,z))dz\theta_{xx}(x,s)\theta_{ttt}(x,s)dxds\\
&-2\int_0^t\int_0^1 \frac{\partial}{\partial s}\int_0^\infty a'(z)F''(\bar\theta_x^s(x,z))dz\theta_{xxt}(x,s)\theta_{ttt}(x,s)dxds \quad t\,\epsilon\,[0,T_0).
\end{aligned}
$$

Here $\theta_{ttt}(\cdot,0)$ and $\theta_{xtt}(\cdot,0)$ are determined from equation (1.1). Making use of (2.1), (2.2), (2.10) (to estimate the first term on the right-hand side of (2.22)), (2.19), (2.20), (2.21), and the Sobolev embedding theorem one can show that

$$
\begin{aligned}
\int_0^1 (\theta_{xtt}^2+\theta_{ttt}^2)(x,t)dx \le \Gamma\{&\Theta+R+\gamma_2^2(t)+\gamma_2^3(t)\\
&+[1+R^{\frac12}+(1+t)(\gamma_2^{\frac12}(t)\\
&+\gamma_2^{\frac{k+1}{2}}(t)]\int_0^t\int_0^1(\theta_{xxx}^2+\theta_{xxt}^2+\theta_{xtt}^2+\theta_{ttt}^2)(x,s)dxds\}\\
&\forall\, t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{2.23}
$$

We differentiate (1.1) twice with respect to t, square the resulting expression, and then integrate over $[0,1]$ to obtain the inequality

$$
\begin{aligned}
&\int_0^1 (\int_0^\infty a'(s)F''(\bar\theta_x^t(x,s))ds)^2\theta_{xxt}^2(x,t)dx\\
&\le 5\int_0^1(\frac{\partial^3}{\partial t^3}(\hat e(\theta(x,t))-\frac{2}{\theta(x,t)}\int_0^\infty a'(s)F(\bar\theta_x^t(x,s))ds))^2dx\\
&+5\int_0^1(\theta_{xx}(x,t)\frac{\partial}{\partial t}\int_0^\infty a'(s)F''(\bar\theta_x^t(x,s))ds)^2dx\\
&+5\int_0^1(\frac{\partial}{\partial t}\int_0^t a'(s)F'''(\bar\theta_x^t(x,s))\theta_{xx}(x,t-s)ds)^2dx\\
&+5\int_0^1(\frac{\partial}{\partial t}\int_0^t a'(s)\frac{\partial}{\partial t}(F''(\bar\theta_x^t(x,s)))\bar\theta_{xx}^t(x,s)ds)^2dx\\
&+5\int_0^1 r_{tt}^2(x,t)dx \qquad \forall\, t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{2.24}
$$

One can show that (2.24) implies

$$
\begin{aligned}
\int_0^1 \theta_{xxt}^2(x,t)dx \le \Gamma\{&R+\gamma_2(t)+\gamma_2^{k+3}(t)\\
&+(1+\gamma_2(t)+\gamma_2^k(t))\int_0^1(\theta_{xtt}^2+\theta_{ttt}^2)(x,t)dx\} \quad \forall\, t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{2.25}
$$

We now differentiate (1.1) once with respect to t and then once with respect to x. We square the result and integrate over $[0,1]$ to get

$$\begin{aligned}
&\int_0^1 \Big(\int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))ds\Big)^2 \theta_{xxx}^2(x,t)dx \\
&\qquad \leq 5\int_0^1 \Big(\frac{\partial^3}{\partial x \partial t^2}(\hat{e}(\theta(x,t)) - \frac{2}{\theta(x,t)}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds)\Big)^2 dx \\
&\qquad + 5\int_0^1 \Big(\theta_{xx}(x,t)\frac{\partial}{\partial x}\int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))ds\Big)^2 dx \\
&\qquad + 5\int_0^1 \Big(\frac{\partial}{\partial x}\int_0^t a'(s)F''(\bar{\theta}_x^t(x,s))\theta_{xx}(x,t-s)ds\Big)^2 dx \\
&\qquad + 5\int_0^1 \Big(\frac{\partial}{\partial x}\int_0^t a'(s)\frac{\partial}{\partial t}F''(\bar{\theta}_x^t(x,s))\bar{\theta}_{xx}^t(x,s)ds\Big)^2 dx \\
&\qquad + 5\int_0^1 r_{xt}^2(x,t)dx \quad \forall\, t \,\epsilon\, [0,T_0).
\end{aligned} \tag{2.26}$$

In order to obtain bounds on the third and fourth terms on the right-hand side of (2.26) we make use of the following observation: we have

$$g(x,t) = g(x,0) + \int_0^t g_t(x,s)ds \quad \forall\, x \,\epsilon\, [0,1],\ t \,\epsilon\, [0,T] \tag{2.27}$$

and hence

$$\int_0^1 g^2(x,t)dx \leq 2\int_0^1 g^2(x,0)dx + 2t\int_0^t \int_0^1 g_t^2(x,s)dxds \quad \forall\, t \,\epsilon\, [0,T] \tag{2.28}$$

for every $T > 0$ and every smooth function $g : [0,1] \times [0,T] \to \mathbb{R}$. Thus we arrive at the inequality

$$\begin{aligned}
\int_0^1 \theta_{xxx}^2(x,t)dx &\leq \Gamma\{R + \gamma_2(t) + \gamma_2^{k+3}(t) \\
&\quad + (1 + \gamma_2(t) + \gamma_2^k(t))\int_0^1 (\theta_{xxt}^2 + \theta_{xtt}^2)(x,t)dx \\
&\quad + t(1 + \gamma_2(t) + \gamma_2^{k+2}(t))\int_0^t \int_0^1 \theta_{xxx}^2(x,s)dxds\} \quad \forall\, t \,\epsilon\, [0,T_0).
\end{aligned} \tag{2.29}$$

We do not give further details of the calculations involved to obtain (2.23), (2.25), (2.29) since they are similar in spirit to calculations described in Section 3 below.

Combining (2.23), (2.25), and (2.29) it is easy to show that there is a constant $N = N(k) > 2$ such that

$$\int_0^1 (\theta_{xxx}^2 + \theta_{xxt}^2 + \theta_{xtt}^2 + \theta_{ttt}^2)(x,t)dx \le \bar{\Gamma}\{\Theta + \Theta^2 + R + R^2 + \gamma_2(T_0) \\ \gamma_2^N(T_0) + [1 + R^{\frac{1}{2}} + R + (1+T_0)(\gamma_2^{\frac{1}{2}}(T_0) + \gamma_2^N(T_0))] \int_0^t \int_0^1 (\theta_{xxx}^2 \\ + \theta_{xxt}^2 + \theta_{xtt}^2 + \theta_{ttt}^2)(x,s)dxds\} \quad \forall t \,\epsilon\, [0, T_0), \tag{2.30}$$

where $\bar{\Gamma}$ is a fixed positive constant independent of r, θ_0, and T_0. Thus Gronwall's inequality implies

$$\gamma_3(T_0) \le \bar{\Gamma}[\Theta + \Theta^2 + R + R^2 + \gamma_2(T_0) + \gamma_2^N(T_0)]exp\{\bar{\Gamma}T_0[1 + R^{\frac{1}{2}} \\ + R + (1+T_0)(\gamma_2^{\frac{1}{2}}(T_0) + \gamma_2^N(T_0))]\}. \tag{2.31}$$

According to Lemma 2.1, if $T_0 < \infty$ then $\gamma_3(T_0) = \infty$ and hence (2.31) leads to the desired conclusion. ∎

In the analysis of (1.1), (1.2), (1.3), (1.4) we make essential use of several properties of strongly positive definite kernels. A function $b \,\epsilon\, L^1_{loc}[0,\infty)$ is said to be **positive definite** if

$$\int_0^t w(s) \int_0^s b(s-z)w(z)dzds \ge 0 \quad \forall t \ge 0, \tag{2.32}$$

for every $w \,\epsilon\, C[0,\infty)$. The kernel b is said to be **strongly positive definite** if there is a constant $c > 0$ such that the mapping $t \mapsto b(t) - ce^{-t}$ is positive definite.

This definition is generally not easy to check directly. One can show that if $b \,\epsilon\, L^1(0,\infty)$, then b is strongly positive definite if and only if there is a constant $c > 0$ such that

$$Re\, \mathcal{L}[b](i\omega) \ge \frac{c}{\omega^2+1} \quad \forall \omega \,\epsilon \mathbb{R}, \tag{2.33}$$

where $\mathcal{L}[\cdot]$ denotes the Laplace transform. It is useful to know that if $b \,\epsilon\, C^2[0,\infty)$ and

$$(-1)^j b^{(j)}(t) \ge 0 \quad \forall t \ge 0, \quad j = 0,1,2, \quad b' \not\equiv 0, \tag{2.34}$$

then b is strongly positive definite. With sufficient regularity, one can obtain information concerning the pointwise behaviour near zero of strongly positive definite functions. In particular, $(1.10)_{1,2}$ imply

$$a(0) > 0, \quad a'(0) < 0. \tag{2.35}$$

This follows easily by expressing $a(0)$ and $a'(0)$ in terms of the Laplace transform of a (cf., e.g. Section 2 of [7]). Condition (2.35) plays an important role in the analysis. See, for example, [11] for more information on strongly positive definite kernels.

In order to obtain certain estimates, we need to solve (1.1) for θ_{xx}. For this purpose we recall that for each $y \in L^1_{loc}[0,\infty)$, the equation

$$a(0)w(t) + \int_0^t a'(t-s)w(s)ds = y(t) \quad t \geq 0 \tag{2.36}$$

has a unique solution $w \in L^1_{loc}[0,\infty)$; this solution is given by

$$w(t) = \frac{1}{a(0)}(y(t) + \int_0^t m(t-s)y(s)ds) \quad t \geq 0, \tag{2.37}$$

where m, the **resolvent kernel of** a', is defined to be the unique solution of the resolvent equation

$$a(0)m(t) + \int_0^t m(t-s)a'(s)ds = -a'(t) \quad t \geq 0. \tag{2.38}$$

Using a Paley-Wiener type argument, $(1.10)_{1,2}$, and properties of strongly positive kernels, we establish the following lemma.

Lemma 2.3: *Assume that* $(1.10)_{1,2}$ *is satisfied. Then the solution* m *to* (2.38) *satisfies* $m' \in L^1(0,\infty)$.

Remark 2.2: *Under assumptions* $(1.10)_{1,2}$ *and* $(1.11)_1$ *one can also show that*

$$m(t) = \frac{a(0)}{\mathcal{L}[a](0)} + M(t) \quad \forall t \geq 0, \tag{2.39}$$

where $M \in L^1(0,\infty)$.

Proof of Lemma 2.3: Define $\Pi := \{\xi \in \mathbf{C} : Re\xi \geq 0\}$. Formally taking Laplace transforms in (2.38) we find that

$$\mathcal{L}[m](\xi) = \frac{-\mathcal{L}[a'](\xi]}{a(0) + \mathcal{L}[a'](\xi)} \quad \forall \xi \in \Pi. \tag{2.40}$$

Recall that $(1.10)_{1,2}$ imply (2.35). Thus, by (2.38) we have

$$\mathcal{L}[m'](\xi) = \frac{a'(0)}{a(0)} - \frac{\xi \mathcal{L}[a'](\xi)}{a(0) + \mathcal{L}[a'](\xi)} \quad \xi \in \Pi. \tag{2.41}$$

After a simple computation we obtain

$$\mathcal{L}[m'](\xi) = \frac{a'(0)}{a(0)} + \frac{a(0)}{\mathcal{L}[a](\xi)} - \xi \quad \xi \in \Pi. \tag{2.42}$$

By (2.33) and the maximum principle for analytic functions $\mathcal{L}[a]$ does not vanish on Π. Hence, by $(2.35)_1$ and (2.33), $\mathcal{L}[m']$ is locally analytic on Π in the sense of Definition 2.1 of [9]. Observe that for ξ near infinity we have

$$\begin{aligned} \mathcal{L}[m'](\xi) &= \frac{-a(0)(\xi\mathcal{L}[a'](\xi) - a'(0)) + a'(0)\mathcal{L}[a'](\xi)}{a(0)(a(0) + \mathcal{L}[a'](\xi))} \\ &= \frac{-a(0)\mathcal{L}[a''](\xi) + a'(0)\mathcal{L}[a'](\xi)}{a(0)(a(0) + \mathcal{L}[a'](\xi))}. \end{aligned} \tag{2.43}$$

Thus $\mathcal{L}[m']$ is locally analytic at infinity and $\mathcal{L}[m'](\infty) = 0$. Therefore, by Proposition 2.3 of [9] $m' \in L^1(0,\infty)$ and the proof is complete. ∎

Before describing our next result we introduce the following notation (which is also used in the next section). For $b \in L^1_{loc}[0,\infty)$ we define

$$Q(w,t,b) := \int_0^t \int_0^1 w(x,s) \int_0^s b(s-z)w(x,z)dzdxds \quad \forall t \in [0,T], \tag{2.44}$$

for every $T > 0$ and every $w \in C([0,T];\, L^2(0,1))$. The result below was motivated by Lemma 2 of [8].

Lemma 2.4: *Assume that* $(1.10)_{1,2}$ *hold. Then there exists a constant* $L > 0$ *such that*

$$\begin{aligned}\int_0^1 w^2(x,t)dx \le L \int_0^1 w^2(x,0)dx + L \int_0^t \int_0^1 w^2(x,s)dxds \\ + L \liminf_{h\downarrow 0} \frac{1}{h^2} Q(\Delta_h w,t,a) \quad \forall t \in [0,T],\end{aligned} \tag{2.45}$$

for every $T > 0$ *and every* $w \in C([0,T];\, L^2(0,1))$ *and consequently, by Lemma* 2.5 *of* [7], *there is a constant* $L^* > 0$ *such that*

$$\begin{aligned}\int_0^1 w^2(x,t)dx \le L^* \int_0^1 w^2(x,0)dx + L^* Q(w,t,a) \\ + L^* \liminf_{h\downarrow 0} \frac{1}{h^2} Q(\Delta_h w,t,a) \quad \forall t \in [0,T],\end{aligned} \tag{2.46}$$

for every $T > 0$ *and every* $w \in C([0,T];\, L^2(0,1))$.

For the proof of Lemma 2.4 it is convenient to introduce the following notation

$$e(t) := e^{-t} \quad \forall t \in [0,\infty). \tag{2.47}$$

In addition, for $T > 0$ and $0 < h < T$, we define the quantity

$$(D_h w)(x,t) := \int_0^t \Delta_h w(x,s)ds \quad \forall t \in [0,T-h], \tag{2.48}$$

for every $w \in C([0,T];\, L^2(0,1))$. We note that

$$(D_h w)(x,t) = \int_t^{t+h} w(x,s)ds - \int_0^h w(x,s)ds \quad t \in [0,T-h]. \tag{2.49}$$

Proof of Lemma 2.4: We first observe that by $(1.10)_{1,2}$ there exists a constant $c > 0$ such that

$$0 \le Q(v,t,e) \le cQ(v,t,a) \quad \forall t \in [0,T], \tag{2.50}$$

for every $T > 0$ and every $v \in C([0,T]; L^2(0,1))$. Let $T > 0, h \in (0,T)$, and $w \in C([0,T]; L^2(0,1))$ be given. Integration by parts (twice) leads to the following identity:

$$\begin{aligned} Q(\Delta_h w,t,e) = \frac{1}{2}\int_0^1 (D_h w)(x,t)^2 dx &+ \int_0^t \int_0^1 (D_h w)(x,s)^2 dxds \\ &- \int_0^1 (D_h w)(x,t) \int_0^t e^{-(t-s)}(D_h w)(x,s) dsdx \\ &- \int_0^t \int_0^1 (D_h w)(x,s) \int_0^s e^{-(s-z)}(D_h w)(x,z) dzdxds. \end{aligned} \tag{2.51}$$

Dividing both sides of (2.51) by h^2 and letting $h \downarrow 0$ we can show that $\lim\limits_{h\downarrow 0} \frac{1}{h^2} Q(\Delta_h w,t,e)$ exists and is given by

$$\begin{aligned} \lim_{h\downarrow 0} \frac{1}{h^2} Q(\Delta_h w,t,e) &= \frac{1}{2}\int_0^1 [w(x,t) - w(x,0)]^2 dx \\ &+ \int_0^t \int_0^1 [w(x,s) - w(x,0)]^2 dxds \\ &- \int_0^1 [w(x,t) - w(x,0)] \int_0^t e^{-(t-s)}[w(x,s) - w(x,0)] dsdx \\ &- \int_0^t \int_0^1 [w(x,s) - w(x,0)] \int_0^s e^{-(s-z)}[w(x,z) - w(x,0)] dzdxds \end{aligned} \tag{2.52}$$

After some simple computations we obtain the following expression for the last two terms on the right-hand side of (2.52)

$$\begin{aligned} &- \int_0^1 [w(x,t) - w(x,0)] \int_0^t e^{-(t-s)}[w(x,s) - w(x,0)] dsdx \\ &\quad = - \int_0^1 w(x,t) \int_0^t e^{-(t-s)} w(x,s) dsdx + \int_0^1 w(x,0) \int_0^t e^{-(t-s)} w(x,s) dsdx \\ &\quad - \int_0^1 w^2(x,0)[1 - e^{-t}] dx + \int_0^1 w(x,t) w(x,0)[1 - e^{-t}] dx, \end{aligned} \tag{2.53}$$

$$
\begin{aligned}
&- \int_0^t \int_0^1 [w(x,s) - w(x,0)] \int_0^s e^{-(s-z)}[w(x,z) - w(x,0)]dzdxds \\
&= -Q(w,t,e) - \int_0^t \int_0^1 w^2(x,0)dxds + \int_0^1 w^2(x,0)[1-e^{-t}]dx \\
&+ 2\int_0^t \int_0^1 w(x,s)w(x,0)dxds - \int_0^t \int_0^1 w(x,s)w(x,0)e^{-s}dxds \\
&- \int_0^t \int_0^1 w(x,s)w(x,0)e^{-(t-s)}dxds.
\end{aligned} \tag{2.54}
$$

Hence, (2.52) implies

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 w^2(x,t)dx = \lim_{h\downarrow 0} \frac{1}{h^2} Q(\Delta_h w, t, e) + Q(w,t,e) - \int_0^t \int_0^1 w^2(x,s)dxds \\
&- \frac{1}{2}\int_0^1 w^2(x,0)dx + \int_0^1 w(x,t)\int_0^t e^{-(t-s)}w(x,s)dsdx + \int_0^1 w(x,t)w(x,0)e^{-t}dx \\
&+ \int_0^t \int_0^1 w(x,s)w(x,0)e^{-s}dxds.
\end{aligned} \tag{2.55}
$$

To complete the proof we use the inequality (2.20); for $\lambda > 0$

$$
\begin{aligned}
&|\int_0^1 w(x,t)\int_0^t e^{-(t-s)}w(x,s)dsdx| \\
&\qquad \le \lambda \int_0^1 w^2(x,t)dx + \frac{1}{4\lambda}\int_0^1 (\int_0^t e^{-(t-s)}w(x,s)ds)^2 dx \\
&\qquad \le \lambda \int_0^1 w^2(x,t)dx + \frac{1}{8\lambda}\int_0^t \int_0^1 w^2(x,s)dxds,
\end{aligned} \tag{2.56}
$$

$$
|\int_0^1 w(x,t)w(x,0)e^{-t}dx| \le \lambda \int_0^1 w^2(x,t)dx + \frac{1}{4\lambda}\int_0^1 w^2(x,0)dx, \tag{2.57}
$$

and similarly

$$
\begin{aligned}
&|\int_0^t \int_0^1 w(x,s)w(x,0)e^{-s}dxds| \\
&\qquad \le \int_0^t \int_0^1 w^2(x,s)dxds + \frac{1}{4}\int_0^t e^{-2s}ds \int_0^1 w^2(x,0)dx \\
&\qquad \le \int_0^t \int_0^1 w^2(x,s)dxds + \frac{1}{8}\int_0^1 w^2(x,0)dx.
\end{aligned} \tag{2.58}
$$

Hence, if $\lambda > 0$ is chosen to be sufficiently small the desired conclusion follows from (2.50).

∎

3. Proof of Theorem 1.1.

We choose $\varepsilon \in (0, \theta^*)$ as in the first paragraph of Section 2. If (1.21) holds with $\delta < \eta/2$, for some $\eta \in (0, \varepsilon)$, then the Sobolev embedding theorem implies

$$|\theta_0(x) - \theta^*|, |\theta_0'(x)| \leq \sqrt{2\Theta_0} < \eta \quad \forall x \in [0,1]. \tag{3.1}$$

Therefore, by Lemmas 2.1 and 2.2, the initial-value problem (1.1), (1.2), (1.3), (1.4) has a unique solution $\theta > 0$ that satisfies

$$\theta, \theta_x, \theta_t, \theta_{xx}, \theta_{xt}, \theta_{tt}, \theta_{xxx}, \theta_{xxt}, \theta_{xtt}, \theta_{ttt} \in C([0, T_0);\ L^2(0,1)) \tag{3.2}$$

and

$$|\theta(x,t) - \theta^*|, |\theta_x(x,t)| < \varepsilon \quad \forall x \in [0,1], t \in [0, T_0) \tag{3.3}$$

on a maximal time interval $[0, T_0), T_0 > 0$. Our aim is to show that if (1.21) holds for $\delta > 0$ sufficiently small, then

$$\begin{aligned} \sup_{t \in [0,T_0)} \int_0^1 ([\theta(x,t) - \theta^*]^2 + \theta_x^2(x,t) + \theta_t^2(x,t) + \theta_{xx}^2(x,t) \\ + \theta_{xt}^2(x,t) + \theta_{tt}^2(x,t))dx < \infty \end{aligned} \tag{3.4}$$

and

$$\sup_{\substack{x \in [0,1] \\ t \in [0,T_0)}} |\theta(x,t) - \theta^*|, \sup_{\substack{x \in [0,1] \\ t \in [0,T_0)}} |\theta_x(x,t)| < \varepsilon \tag{3.5}$$

and hence $T_0 = \infty$ (by Lemma 2.2). For this purpose it is convenient to introduce the quantities

$$\begin{aligned} \mathcal{E}(t) := \sup_{s \in [0,t]} \int_0^1 & ([\theta(x,s) - \theta^*]^2 + \theta_x^2(x,s) + \theta_t^2(x,s) + \theta_{xx}^2(x,s) \\ & + \theta_{xt}^2(x,s) + \theta_{tt}^2(x,s))dx \\ & + \int_0^t \int_0^1 ([\theta(x,s) - \theta^*]^2 + \theta_x^2(x,s) + \theta_t^2(x,s) + \theta_{xx}^2(x,s) \\ & + \theta_{xt}^2(x,s) + \theta_{tt}^2(x,s))dxds \qquad t \in [0, T_0) \end{aligned} \tag{3.6}$$

and

$$\nu(t) := \sup_{\substack{x\in[0,1] \\ s\in[0,t]}} ([\theta(x,s)-\theta^*]^2 + \theta_x^2(x,s) + \theta_t^2(x,s))^{\frac{1}{2}} \\ + (\int_0^t (\sup_{x\in[0,1]} |\theta_x(x,s)|)^2 ds)^{\frac{1}{2}} \quad t \in [0,T_0). \tag{3.7}$$

Equation (1.1) can be rewritten as follows

$$\begin{aligned} &\hat{e}'(\theta^*)\theta_t(x,t) - F''(0)\int_0^t a(t-s)\theta_{xx}(x,s)ds \\ &= -(\hat{e}'(\theta(x,t)) - \hat{e}'(\theta^*))\theta_t(x,t) - \int_0^t \theta_{xx}(x,s)\int_{t-s}^\infty a'(z)(F''(\bar{\theta}_x^t(x,z)) \\ &- F''(0))dzds - \frac{2}{\theta(x,t)^2}\theta_t(x,t)\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds \\ &+ \frac{2}{\theta(x,t)}\theta_x(x,t)\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))ds \\ &- \frac{2}{\theta(x,t)}\int_0^t a'(s)F'(\bar{\theta}_x^t(x,s))\theta_x(x,t-s)ds + r(x,t) \\ &\qquad x \in [0,1], t \in [0,T_0). \end{aligned} \tag{3.8}$$

In the derivation of this equation from (1.1) we make use of (1.2) and (1.8). The second terms on both sides of (3.8) are obtained through the following computation

$$\begin{aligned} \int_0^\infty a'(s)F''(\bar{\theta}_x^t(x,s))\bar{\theta}_{xx}^t(x,s)ds &= \int_0^\infty \int_{t-s}^t a'(s)F''(\bar{\theta}_x^t(x,s))\theta_{xx}(x,z)dzds \\ &= \int_0^t \theta_{xx}(x,z)\int_{t-z}^\infty a'(s)F''(\bar{\theta}_x^t(x,s))dsdz. \end{aligned} \tag{3.9}$$

The aim of the computations that follow is to establish the inequality (3.40) below; to do so we employ energy methods. We use two main types of estimates in this argument:

(i) estimates derived directly from energy integrals;

(ii) additional estimates obtained from equation (3.8) through the use of inverse Volterra operators.

In our energy integrals, the left-hand side of (3.8) will lead to positive definite contributions and the right-hand side will lead to terms that are small provided the solution is near equilibrium. We make essential use of Lemma 2.4 in the estimates of type (i); in addition, in order to estimate the energy integral of highest order one must exploit compatibility of our constitutive relations (0.19), (0.22) with thermodynamics, i.e. we make use of the fact that a solution of (1.1), (1.2), (1.3), (1.4) satisfies the entropy inequality (0.2). (See Remark 2.1 for further details.) Lemma 2.3 plays an important role in the estimates of type (ii). A reader who is unfamiliar with energy methods and seeks further motivation for our computations may wish to look at the argument following (3.40) before reading the derivation of (3.40).

In the numerous estimations that follow we make frequent use of the inequalities (2.19), (2.20), and (2.21). We use Γ to denote a (possibly large) positive generic constant which is independent of θ_0, r, and T_0.

To obtain our first energy integral we multiply equation (3.8) by $(\theta-\theta^*)$ and integrate over $[0,1]\times[0,t], t\,\epsilon\,[0,T_0)$. After integration by parts we find that

$$
\begin{aligned}
&\frac{1}{2}\hat{e}'(\theta^*)\int_0^1[\theta(x,t)-\theta^*]^2dx+F''(0)Q(\theta_x,t,a)\\
&=\frac{1}{2}\hat{e}'(\theta^*)\int_0^1[\theta_0(x)-\theta^*]^2dx\\
&\quad+\int_0^t\int_0^1[\theta(x,s)-\theta^*]\{-(\hat{e}'(\theta(x,s))-\hat{e}'(\theta^*))\theta_t(x,s)\\
&\quad-\int_0^s\theta_{xx}(x,y)\int_{s-y}^\infty a'(z)(F''(\bar{\theta}_x^s(x,z))-F''(0))dzdy\\
&-\frac{2}{\theta(x,s)^2}\theta_t(x,s)\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz\\
&+\frac{2}{\theta(x,s)}\theta_x(x,s)\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz\\
&-\frac{2}{\theta(x,s)}\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dz+r(x,s)\}dxds\quad\forall\,t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{3.10}
$$

We next differentiate (3.8) with respect to t:

$$
\begin{aligned}
&\hat{e}'(\theta^*)\theta_{tt}(x,t) - F''(0)a(0)\theta_{xx}(x,t) - F''(0)\int_0^t a'(t-s)\theta_{xx}(x,s)ds \\
&= \frac{\partial}{\partial t}\{-(\hat{e}'(\theta(x,t)) - \hat{e}'(\theta^*))\theta_t(x,t) \\
&- \int_0^t \theta_{xx}(x,s)\int_{t-s}^{\infty} a'(z)(F''(\bar{\theta}_x^t(x,z)) - F''(0))dzds \\
&- \frac{2}{\theta(x,t)^2}\theta_t(x,t)\int_0^{\infty} a'(s)F(\bar{\theta}_x^t(x,s))ds + \frac{2}{\theta(x,t)}\theta_x(x,t)\int_0^{\infty} a'(s)F'(\bar{\theta}_x^t(x,s))ds \\
&- \frac{2}{\theta(x,t)}\int_0^t a'(s)F'(\bar{\theta}_x^t(x,s))\theta_x(x,t-s)ds + r(x,t)\} \\
&\qquad x\,\epsilon\,[0,1], t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{3.11}
$$

Multiplying this equation by θ_t and integrating over $[0,1]\times[0,t], t\,\epsilon\,[0,T_0)$ we obtain the following expression:

$$
\begin{aligned}
&\frac{1}{2}\hat{e}'(\theta^*)\int_0^1 \theta_t^2(x,t)dx + F''(0)Q(\theta_{xt},t,a) \\
&= F''(0)\int_0^t\int_0^1 a(s)\theta_0''(x)\theta_t(x,s)dxds + \frac{1}{2}\hat{e}'(\theta^*)\int_0^1 \theta_t^2(x,0)dx \\
&+ \int_0^t\int_0^1 \theta_t(x,s)\frac{\partial}{\partial s}\{-(\hat{e}'(\theta(x,s)) - \hat{e}'(\theta^*))\theta_t(x,s) \\
&- \int_0^s \theta_{xx}(x,y)\int_{s-y}^{\infty} a'(z)(F''(\bar{\theta}_x^s(x,z)) - F''(0))dzdy \\
&- \frac{2}{\theta(x,s)^2}\theta_t(x,s)\int_0^{\infty} a'(z)F(\bar{\theta}_x^s(x,z))dz \\
&+ \frac{2}{\theta(x,s)}\theta_x(x,s)\int_0^{\infty} a'(z)F'(\bar{\theta}_x^s(x,z))dz \\
&- \frac{2}{\theta(x,s)}\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dz + r(x,s)\}dxds \quad \forall\, t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{3.12}
$$

We note that according to equation (3.8) we have

$$
\theta_t(x,0) = \frac{1}{\hat{e}'(\theta_0(x))}r(x,0) \quad x\,\epsilon\,[0,1]. \tag{3.13}
$$

Differentiation of equation (3.11) with respect to t yields (after integrating several terms by parts)

$$
\begin{aligned}
&\hat{e}'(\theta^*)\theta_{ttt}(x,t) - F''(0)a(0)\theta_{xxt}(x,t) - F''(0)\int_0^t a'(t-s)\theta_{xxt}(x,s)ds \\
&= F''(0)a'(t)\theta_0''(x) + \frac{\partial^2}{\partial t^2}\{-(\hat{e}'(\theta(x,t)) - \hat{e}'(\theta^*))\theta_t(x,t) \\
&- \frac{2}{\theta(x,t)}\int_0^t a'(s)F'(\bar{\theta}_x^t(x,s))\theta_x(x,t-s)ds + r(x,t)\} \\
&+ \frac{\partial}{\partial t}\{\int_0^t \theta_{xx}(x,t-s)a'(s)(F''(\bar{\theta}_x^t(x,s)) - F''(0))ds \\
&+ \int_0^t \theta_{xx}(x,s)\int_{t-s}^t a'(z)F'''(\bar{\theta}_x^t(x,z))\theta_x(x,t-z)dzds\} \\
&- \theta_{xx}(x,t)\frac{\partial}{\partial t}\int_0^\infty a'(s)(F''(\bar{\theta}_x^t(x,s)) - F''(0))ds \\
&- \theta_x(x,t)\frac{\partial}{\partial t}\int_0^t \theta_{xx}(x,s)\int_{t-s}^\infty a'(z)F'''(\bar{\theta}_x^t(x,z))dzds \\
&- \frac{\partial}{\partial x}(\theta_{xt}(x,t)\int_0^\infty a'(s)(F''(\bar{\theta}_x^t(x,s)) - F''(0))ds) \\
&- \theta_t(x,t)\frac{\partial^2}{\partial t^2}(\frac{2}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds) \\
&- \theta_{tt}(x,t)\frac{\partial}{\partial t}(\frac{4}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds) \\
&- \theta_{ttt}(x,s)\frac{2}{\theta(x,t)^2}\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))ds \\
&+ \theta_x(x,t)\frac{\partial^2}{\partial t^2}(\frac{2}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))ds) \\
&+ \theta_{xt}(x,t)\frac{\partial}{\partial t}(\frac{4}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))ds) \\
&+ \theta_{xtt}(x,t)\frac{2}{\theta(x,t)}\int_0^\infty a'(s)F'(\bar{\theta}_x^t(x,s))ds \quad x\in[0,1], t\in[0,T_0).
\end{aligned}
\tag{3.14}
$$

In analogy with the previous calculation, we multiply (3.14) by θ_{tt} and integrate over $[0,1]\times[0,t], t\in[0,T_0)$. The resulting relation is

$$
\begin{aligned}
&\frac{1}{2}\hat{e}'(\theta*)\int_0^1 \theta_{tt}^2(x,t)dx + F''(0)Q(\theta_{xtt},t,a)\\
&= F''(0)\int_0^t\int_0^1 a'(s)\theta_0''(x)\theta_{tt}(x,s)dxds - F''(0)\int_0^1 a(t)\theta_{xt}(x,0)\theta_{xt}(x,t)dx\\
&+ F''(0)a(0)\int_0^1 \theta_{xt}^2(x,0)dx + F''(0)\int_0^t\int_0^1 a'(s)\theta_{xt}(x,0)\theta_{xt}(x,s)dxds\\
&+ \frac{1}{2}\hat{e}'(\theta^*)\int_0^1 \theta_{tt}^2(x,0)dx + \int_0^t\int_0^1 \theta_{tt}(x,s)\frac{\partial^2}{\partial s^2}\{-(\hat{e}'(\theta(x,s)) - \hat{e}'(\theta^*))\theta_t(x,s)\\
&- \frac{2}{\theta(x,s)}\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dz + r(x,s)\}dxds\\
&+ \int_0^t\int_0^1 \theta_{tt}(x,s)\frac{\partial}{\partial s}\{\int_0^s \theta_{xx}(x,s-z)a'(z)(F''(\bar{\theta}_x^s(x,z)) - F''(0))dz\\
&+ \int_0^s \theta_{xx}(x,y)\int_{s-y}^s a'(z)F'''(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dzdy\}dxds\\
&- \int_0^t\int_0^1 \theta_{tt}(x,s)\theta_{xx}(x,s)\frac{\partial}{\partial s}\int_0^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) - F''(0))dzdxds\\
&- \int_0^t\int_0^1 \theta_{tt}(x,s)\theta_x(x,s)\frac{\partial}{\partial s}\int_0^s \theta_{xx}(x,y)\int_{s-y}^\infty a'(z)F'''(\bar{\theta}_x^s(x,z))dzdydxds\\
&+ \frac{1}{2}\int_0^1 \theta_{xt}^2(x,t)\int_0^\infty a'(s)(F''(\bar{\theta}_x^t(x,s)) - F''(0))dsdx \qquad (3.15)\\
&- \frac{1}{2}\int_0^t\int_0^1 \theta_{xt}^2(x,s)\frac{\partial}{\partial s}\int_0^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) - F''(0))dzdxds\\
&- \int_0^t\int_0^1 \theta_{tt}(x,s)\theta_t(x,s)\frac{\partial^2}{\partial s^2}(\frac{2}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz)dxds\\
&- \int_0^t\int_0^1 \theta_{tt}^2(x,s)\frac{\partial}{\partial s}(\frac{4}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz)dxds\\
&- \int_0^1 \frac{1}{\theta(x,t)^2}\theta_{tt}^2(x,t)\int_0^\infty a'(s)F(\bar{\theta}_x^t(x,s))dsdx\\
&+ \int_0^t\int_0^1 \theta_{tt}^2(x,s)\frac{\partial}{\partial s}(\frac{1}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz)dxds\\
&+ \int_0^t\int_0^1 \theta_{tt}(x,s)\theta_x(x,s)\frac{\partial^2}{\partial s^2}(\frac{2}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz)dxds\\
&+ \int_0^t\int_0^1 \theta_{tt}(x,s)\theta_{xt}(x,s)\frac{\partial}{\partial s}(\frac{4}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz)dxds\\
&- \int_0^t\int_0^1 \theta_{tt}^2(x,s)\frac{\partial}{\partial x}(\frac{1}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz)dxds \quad \forall\, t \in [0,T_0).
\end{aligned}
$$

We note that (3.13) implies

$$\theta_{xt}(x,0) = -\frac{\hat{e}''(\theta_0(x))}{\hat{e}'(\theta_0(x))}\theta_0'(x)\theta_t(x,0) + \frac{1}{\hat{e}'(\theta_0(x))}r_x(x,0) \quad x \in [0,1] \tag{3.16}$$

and from (3.11) we have

$$\begin{aligned}\theta_{tt}(x,0) = &\frac{F''(0)a(0)}{\hat{e}'(\theta_0(x))}\theta_0''(x) - \frac{\hat{e}''(\theta_0(x))}{\hat{e}'(\theta_0(x))}\theta_t^2(x,0) \\ &- \frac{2F''(0)a(0)}{\theta_0(x)\hat{e}'(\theta_0(x))}\theta_0'(x)^2 + \frac{1}{\hat{e}'(\theta_0(x))}r_t(x,0) \quad x \in [0,1].\end{aligned} \tag{3.17}$$

We add (3.10), (3.12), and (3.15) and make use of Lemma 2.4 to obtain a lower bound on the left-hand side of the resulting identity. We then make some routine estimations to derive the inequality

$$\begin{aligned}&\int_0^1 ([\theta(x,t) - \theta^*]^2 + \theta_x^2(x,t) + \theta_t^2(x,t) + \theta_{xt}^2(x,t) + \theta_{tt}^2(x,t))dx \\ &\quad + \int_0^t \int_0^1 (\theta_x^2(x,s) + \theta_{xt}^2(x,s))dxds \le \Gamma\{\Theta_0 + R_0\} + \Gamma\{\sqrt{\Theta_0} + \sqrt{R_0}\}\sqrt{\mathcal{E}(t)} \\ &\quad + \Gamma\sqrt{R_0}\mathcal{E}(t) + \Gamma\{\nu(t) + \nu^{k+2}(t)\}\mathcal{E}(t) \\ &\quad \forall t \in [0,T_0).\end{aligned} \tag{3.18}$$

In order to give an indication of how (3.18) was derived we show detailed estimations of certain typical terms of (3.10), (3.12), and (3.15) as follows. Many of the terms can be estimated in a simple way, for instance

$$\begin{aligned}&\left|\int_0^t \int_0^1 \hat{e}'''(\theta(x,s))\theta_t^2(x,s)\theta_{tt}(x,s)dxds\right| \\ &\quad \le \sup_{\substack{x\in[0,1]\\ s\in[0,t]}} |\hat{e}''(\theta(x,s))\theta_t(x,s)| \int_0^t \int_0^1 |\theta_t(x,s)\theta_{tt}(x,s)|dxds \\ &\quad \le \Gamma\nu(t)\int_0^t \int_0^1 |\theta_{tt}(x,s)\theta_t(x,s)|dxds \\ &\quad \le \Gamma\nu(t)\int_0^t \int_0^1 (\theta_{tt}^2(x,s) + \theta_t^2(x,s))dxds \\ &\quad \le \Gamma\nu(t)\mathcal{E}(t) \qquad \forall t \in [0,T_0)\end{aligned} \tag{3.19}$$

or

$$
\begin{aligned}
&|F''(0)\int_0^t\int_0^1 a'(s)\theta_0''(x)\theta_{tt}(x,s)dxds| \\
&\qquad\le F''(0)(\int_0^t\int_0^1 a'(s)^2\theta_0''(x)^2dxds)^{\frac{1}{2}}(\int_0^t\int_0^1 \theta_{tt}^2(x,s)dxds)^{\frac{1}{2}} \\
&\qquad\le F''(0)(\int_0^\infty a'(s)^2ds)^{\frac{1}{2}}(\int_0^1 \theta_0''(x)^2dx)^{\frac{1}{2}}\sqrt{\mathcal{E}(t)} \\
&\qquad\le \Gamma\sqrt{\Theta_0}\sqrt{\mathcal{E}(t)} \qquad \forall\, t\in[0,T_0).
\end{aligned}
\tag{3.20}
$$

Some of the terms must be rewritten carefully before they are estimated: e.g. the term estimated in (3.25) below arises from

$$
\int_0^t\int_0^1 \theta_{tt}(x,s)\frac{\partial^2}{\partial s^2}(\frac{2}{\theta(x,s)}\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dz)dxds \tag{3.21}
$$

which appears on the right-hand side of (3.15). We first differentiate the integral appearing in the integrand of (3.21) once with respect to s and then make the following change of variable

$$
\begin{aligned}
&\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_{xt}(x,s-z)dz \\
&\qquad\qquad = \int_0^s a'(s-\zeta)F'(\bar{\theta}_x^s(x,s-\zeta))\theta_{xt}(x,\zeta)d\zeta.
\end{aligned}
\tag{3.22}
$$

We next differentiate the right-hand side of (3.22) with respect to s and then repeat the same change of variable to obtain the integral estimated in (3.25). We note that a similar procedure is used when differentiating terms of the form

$$
\int_{s-y}^\infty a'(z)F'''(\bar{\theta}_x^s(x,z))dz \tag{3.23}
$$

with respect to s; the change of variable in this case takes the form

$$
\int_{s-y}^\infty a'(z)F'''(\bar{\theta}_x^s(x,z))dz = \int_{-\infty}^y a'(s-\zeta)F'''(\bar{\theta}_x^s(x,s-\zeta))d\zeta. \tag{3.24}
$$

We now continue to show some typical calculations. The computations below are more involved than those used in (3.19) and (3.20): we obtain a bound on a term appearing on the right-hand side of (3.15)

$$
\begin{aligned}
&|\int_0^t \int_0^1 \theta_{tt}(x,s)\frac{2}{\theta(x,s)} \int_0^s a'(z)F''(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)\theta_{xt}(x,s-z)dzdxds| \\
&\leq \Gamma \sup_{\substack{x\epsilon[0,1]\\ s\epsilon[0,t]}} |\theta_x(x,s)| \int_0^t \int_0^1 |\theta_{tt}(x,s)| \int_0^s |a'(z)|(F''(0) \\
&\quad + |F''(\bar{\theta}_x^s(x,z)) - F''(0)|)\,|\theta_{xt}(x,s-z)|dzdxds \\
&\leq \Gamma\nu(t) \int_0^t \int_0^1 |\theta_{tt}(x,s)| \int_0^s |a'(z)|(F''(0) + K(|\bar{\theta}_x^s(x,z)| \\
&\quad + |\bar{\theta}_x^s(x,z)|^k))|\theta_{xt}(x,s-z)|dzdxds \\
&\leq \Gamma\nu(t) \int_0^t \int_0^1 |\theta_{tt}(x,s)| \int_0^s |a'(z)|(F''(0) + K[\sqrt{z}(\int_{s-z}^s \theta_x^2(x,\xi)d\xi)^{\frac{1}{2}} \\
&\quad + (\sqrt{z})^k(\int_{s-z}^s \theta_x^2(x,\xi)d\xi)^{\frac{k}{2}}])|\theta_{xt}(x,s-z)|dzdxds \qquad\qquad (3.25)\\
&\leq \Gamma\nu(t)(\int_0^t \int_0^1 \theta_{tt}^2(x,s)dxds)^{\frac{1}{2}}(\int_0^t \int_0^1 (\int_0^s |a'(z)|[1 + \nu(t)\sqrt{z} \\
&\quad + \nu^k(t)(\sqrt{z})^k]|\theta_{xt}(x,s-z)|dz)^2 dxds)^{\frac{1}{2}} \\
&\leq \Gamma\nu(t)\sqrt{\mathcal{E}(t)}(\int_0^t \int_0^1 \theta_{xt}^2(x,s)dxds)^{\frac{1}{2}}(\int_0^\infty |a'(z)|dz + \nu(t)\int_0^\infty |a'(z)|\sqrt{z}dz \\
&\quad + \nu^k(t)\int_0^\infty |a'(z)|(\sqrt{z})^k dz) \\
&\leq \Gamma\{\nu(t) + \nu^2(t) + \nu^{k+1}(t)\}\mathcal{E}(t) \leq \Gamma\{\nu(t) + \nu^{k+1}(t)\}\mathcal{E}(t) \\
&\qquad\qquad \forall\, t\,\epsilon\,[0,T_0)
\end{aligned}
$$

and from (3.10) we estimate the following term

$$
\begin{aligned}
&|\int_0^t\int_0^1[\theta(x,s)-\theta^*]\int_0^s\theta_{xx}(x,y)\int_{s-y}^\infty a'(z)(F''(\bar\theta_x^s(x,z))-F''(0))dzdydxds| \\
&\quad\le K(\nu(t)+\nu^k(t))\int_0^t\int_0^1|\theta(x,s)-\theta^*|\int_0^s|\theta_{xx}(x,y)|\int_{s-y}^\infty|a'(z)|(\sqrt{z} \\
&\qquad+(\sqrt{z})^k)dzdydxds \\
&\quad\le K(\nu(t)+\nu^k(t))\sqrt{\mathcal{E}(t)}(\int_0^t\int_0^1(\int_0^s|\theta_{xx}(x,y)|\int_{s-y}^\infty|a'(z)|(\sqrt{z} \\
&\qquad+(\sqrt{z})^k)dzdy)^2dxds)^{\frac{1}{2}} \\
&\quad\le K(\nu(t)+\nu^k(t))\mathcal{E}(t)\int_0^\infty\int_s^\infty|a'(z)|(\sqrt{z}+(\sqrt{z})^k)dzds \\
&\quad\le \Gamma\{\nu(t)+\nu^k(t)\}\mathcal{E}(t) \qquad \forall\, t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{3.26}
$$

The rest of the terms on the right-hand side of (3.10), (3.12), and (3.15), except for the last term in (3.15), are handled in a similar fashion to (3.19), (3.20), (3.25), and (3.26). The last term on the right-hand side of (3.15) is first estimated from above, making use of compatibility with thermodynamics, i.e. we utilize the entropy inequality (0.2) in the estimation below: by (2.10) we have

$$
\begin{aligned}
&-\int_0^t\int_0^1\theta_{tt}^2(x,s)\frac{\partial}{\partial x}(\frac{1}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar\theta_x^s(x,z))dz)dxds \\
&\quad\le\int_0^t\int_0^1\theta_{tt}^2(x,s)\{\frac{\partial}{\partial s}(-\hat\psi'(\theta(x,s))-\frac{1}{\theta(x,s)^2}\int_0^\infty a'(z)F(\bar\theta_x^s(x,z))dz) \\
&\qquad-\frac{r(x,s)}{\theta(x,s)}\}dxds \quad \forall\, t\,\epsilon\,[0,T_0),
\end{aligned}
\tag{3.27}
$$

where $\hat\psi''\,\epsilon\, C(0,\infty)$ (see Remark 2.1). Thus, it can be shown that

$$
\begin{aligned}
&-\int_0^t\int_0^1\theta_{tt}^2(x,s)\frac{\partial}{\partial x}(\frac{1}{\theta(x,s)}\int_0^\infty a'(z)F'(\bar\theta_x^s(x,z))dz)dxds \\
&\qquad\le\Gamma\{\nu(t)+\nu^{k+1}(t)\}\mathcal{E}(t)+\Gamma\sqrt{R_0}\mathcal{E}(t) \quad \forall\, t\,\epsilon\,[0,T_0).
\end{aligned}
\tag{3.28}
$$

Additonal estimates are derived directly from equation (3.8) in the following manner. In order to obtain a temporal$-L^2$ estimate for θ_t we first multiply (3.8) by θ_{xx} and integrate the resulting identity over $[0,1] \times [0,t]$, $t \in [0,T_0)$. We arrive at the relation

$$
\begin{aligned}
&\frac{1}{2}\hat{e}'(\theta^*)\int_0^1 \theta_x^2(x,t)dx + F''(0)Q(\theta_{xx},t,a) \\
&\quad = \frac{1}{2}\hat{e}'(\theta^*)\int_0^1 \theta_0'(x)^2 dx \\
&\quad + \int_0^t\int_0^1 \theta_{xx}(x,s)\{(\hat{e}'(\theta(x,s)) - \hat{e}'(\theta^*))\theta_t(x,s) \\
&\qquad + \int_0^s \theta_{xx}(x,y)\int_{s-y}^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) - F''(0))dzdy \\
&\qquad + \frac{2}{\theta(x,s)^2}\theta_t(x,s)\int_0^\infty a'(z)F(\bar{\theta}_x^s(x,z))dz \\
&\quad - \frac{2}{\theta(x,s)}\theta_x(x,s)\int_0^\infty a'(z)F'(\bar{\theta}_x^s(x,z))dz \\
&\qquad + \frac{2}{\theta(x,s)}\int_0^s a'(z)F'(\bar{\theta}_x^s(x,z))\theta_x(x,s-z)dz - r(x,s)\}dxds \\
&\qquad\qquad \forall\, t \in [0,T_0).
\end{aligned}
\tag{3.29}
$$

This relation leads to the inequality

$$
\begin{aligned}
&Q(\theta_{xx},t,a) \le \Gamma\Theta_0 + \Gamma\sqrt{R_0}\sqrt{\mathcal{E}(t)} + \Gamma\{\nu(t) + \nu^k(t)\}\mathcal{E}(t) \\
&\qquad\qquad\qquad\qquad \forall\, t \in [0,T_0).
\end{aligned}
\tag{3.30}
$$

We now square (3.8) and integrate over $[0,1] \times [0,t]$, $t \in [0,T_0)$. Using (1.10), Lemma 4.2 of [12], and (3.30) we arrive at the estimate

$$
\begin{aligned}
\int_0^t\int_0^1 \theta_t^2(x,s)dxds &\le \Gamma\{\Theta_0 + R_0\} + \Gamma\sqrt{R_0}\sqrt{\mathcal{E}(t)} \\
&\quad + \Gamma\{\nu(t) + \nu^{2k}(t)\}\mathcal{E}(t) \quad \forall\, t \in [0,T_0).
\end{aligned}
\tag{3.31}
$$

Equation (3.11) can be written as

$$
\begin{aligned}
&\hat{e}'(\theta^*)\theta_{tt}(x,t) - F''(0)a(0)\theta_{xx}(x,t) - F''(0)\int_0^t a'(t-s)\theta_{xx}(x,s)ds = G_t(x,t) \\
&\qquad\qquad\qquad\qquad x \in [0,1],\ t \in [0,T_0),
\end{aligned}
\tag{3.32}
$$

where $G(x,t)$ denotes the right-hand side of (3.8).

Solving for θ_{xx} in terms of θ_{tt} and G_t (see (2.36) and (2.37)) we get

$$\begin{aligned}
&-F''(0)a(0)\theta_{xx}(x,t) \\
&= G_t(x,t) - \hat{e}'(\theta^*)\theta_{tt}(x,t) + \int_0^t m(t-s)[G_t(x,s) - \hat{e}'(\theta^*)\theta_{tt}(x,s)]ds \qquad (3.33) \\
&\qquad x \in [0,1],\ t \in [0,T_0),
\end{aligned}$$

where m is the resolvent of a' (see (2.38)). We note that by (3.8) $G(x,0) - \hat{e}'(\theta^*)\theta_t(x,0) = 0$ for all $x \in [0,1]$, and by (2.38) $m(0) = -a'(0)/a(0)$. Thus, after integrating the last term on the right-hand side of (3.33) by parts we arrive at the expression

$$\begin{aligned}
&-F''(0)a(0)\theta_{xx}(x,t) \\
&= G_t(x,t) - \hat{e}'(\theta*)\theta_{tt}(x,t) - \frac{a'(0)}{a(0)}[G(x,t) - \hat{e}'(\theta^*)\theta_t(x,t)] \qquad (3.34) \\
&\quad + \int_0^t m'(t-s)[G(x,s) - \hat{e}'(\theta^*)\theta_t(x,s)]ds \quad x \in [0,1],\ t \in [0,T_0).
\end{aligned}$$

We now square (3.34) and integrate over $[0,1]$. By (3.18) and Lemma 2.3 we have

$$\begin{aligned}
\int_0^1 \theta_{xx}^2(x,t)dx &\le \Gamma\{\Theta_0 + R_0\} + \Gamma\{\sqrt{\Theta_0} + \sqrt{R_0}\}\sqrt{\mathcal{E}(t)} \qquad (3.35) \\
&\quad + \Gamma\sqrt{R_0}\mathcal{E}(t) + \Gamma\{\nu(t) + \nu^{2k+2}(t)\}\mathcal{E}(t) \quad \forall t \in [0,T_0).
\end{aligned}$$

To obtain a temporal$-L^2$ bound on θ_{tt} we multiply (3.34) by θ_{tt} and integrate over $[0,1] \times [0,t]$, $t \in [0,T_0)$. We note that

$$\begin{aligned}
\int_0^t \int_0^1 \theta_{tt}(x,s)\theta_{xx}(x,s)dxds &= -\int_0^t \int_0^1 \theta_x(x,s)\theta_{xtt}(x,s)dxds \\
&= -\int_0^1 \theta_x(x,t)\theta_{xt}(x,t)dx + \int_0^1 \theta_0'(x)\theta_{xt}(x,0)dx \qquad (3.36) \\
&\quad + \int_0^t \int_0^1 \theta_{xt}^2(x,s)dxds \quad t \in [0,T_0).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int_0^t \int_0^1 \theta_{tt}^2(x,s)dxds &\le \Gamma\{\Theta_0 + R_0\} + \Gamma\{\sqrt{\Theta_0} + \sqrt{R_0}\}\sqrt{\mathcal{E}(t)} \qquad (3.37) \\
&\quad + \Gamma\sqrt{R_0}\mathcal{E}(t) + \Gamma\{\nu(t) + \nu^{k+2}(t)\}\mathcal{E}(t) \quad \forall t \in [0,T_0).
\end{aligned}$$

(Here, we make crucial use of the inequality (2.20).) We now square (3.34) and integrate over $[0,1] \times [0,t]$, $t \in [0,T_0)$ using (3.18), (3.31), (3.37), and Lemma 2.3 to obtain the following estimate

$$\begin{aligned}\int_0^t \int_0^1 \theta_{xx}^2(x,s)dxds \leq \Gamma\{\Theta_0 + R_0\} + \Gamma\{\sqrt{\Theta_0} + \sqrt{R_0}\}\sqrt{\mathcal{E}(t)} \\ + \Gamma\sqrt{R_0}\mathcal{E}(t) + \Gamma\{\nu(t) + \nu^{2k+2}(t)\}\mathcal{E}(t) \quad \forall\, t \in [0,T_0).\end{aligned} \tag{3.38}$$

Observe that by Poincaré's inequality there is a constant $c > 0$ such that

$$\int_0^t \int_0^1 [\theta(x,s) - \theta^*]^2 dxds \leq c \int_0^t \int_0^1 \theta_x^2(x,s)dxds \quad \forall\, t \in [0,T_0). \tag{3.39}$$

It follows from (3.18), (3.31), (3.35), (3.37), (3.38), and (3.39) that

$$\begin{aligned}\mathcal{E}(t) \leq \Gamma\{\Theta_0 + R_0\} + \Gamma\{\sqrt{\Theta_0} + \sqrt{R_0}\}\sqrt{\mathcal{E}(t)} + \Gamma\sqrt{R_0}\mathcal{E}(t) \\ + \Gamma\{\nu(t) + \nu^{2k+2}(t)\}\mathcal{E}(t) \quad \forall\, t \in [0,T_0).\end{aligned} \tag{3.40}$$

Using (2.20), (3.40) yields

$$\begin{aligned}\mathcal{E}(t) \leq \bar{\Gamma}\{\Theta_0 + R_0\} + \bar{\Gamma}\sqrt{R_0}\mathcal{E}(t) + \Gamma\{\nu(t) + \nu^{2k+2}(t)\}\mathcal{E}(t) \\ \forall\, t \in [0,T_0),\end{aligned} \tag{3.41}$$

where $\bar{\Gamma}$ denotes a fixed positive constant which is independent of θ_0, r, and T_0. We choose $\overline{\mathcal{E}}, \delta > 0$ such that

$$\overline{\mathcal{E}} < \varepsilon^2, \quad \bar{\Gamma}\{\sqrt{2\overline{\mathcal{E}}} + (\sqrt{2\overline{\mathcal{E}}})^{2k+2}\} \leq \frac{1}{6}, \quad \bar{\Gamma}\delta^2 \leq \frac{1}{6}\overline{\mathcal{E}}, \quad \bar{\Gamma}\delta \leq \frac{1}{6}, \tag{3.42}$$

and

$$\delta < \frac{1}{2}\eta \tag{3.43}$$

for some $\eta \in (0,\varepsilon)$.

Suppose now that (1.21) holds for the above choice of δ. By the Sobolev embedding theorem

$$\nu(t) \le \sqrt{2\mathcal{E}(t)} \qquad \forall\, t \in [0, T_0). \tag{3.44}$$

Thus, it follows from (3.41) that for any $t \in [0, T_0)$ with $\mathcal{E}(t) \le \overline{\mathcal{E}}$, we actually have $\mathcal{E}(t) \le \frac{1}{2}\overline{\mathcal{E}}$. Hence, by continuity, if $\mathcal{E}(0) \le \frac{1}{2}\overline{\mathcal{E}}$ then

$$\mathcal{E}(t) \le \frac{1}{2}\overline{\mathcal{E}} \qquad \forall\, t \in [0, T_0). \tag{3.45}$$

It is possible to choose a smaller $\delta > 0$ (if necessary) so that (1.21) implies $\mathcal{E}(0) \le \frac{1}{2}\overline{\mathcal{E}}$. Consequently, for $\delta > 0$ small enough, (3.45) holds; moreover, by the Sobolev embedding theorem

$$\sup_{\substack{x \in [0,1] \\ t \in [0, T_0)}} |\theta(x,t) - \theta^*|, \ \sup_{\substack{x \in [0,1] \\ t \in [0, T_0)}} |\theta_x(x,t)| \le \sqrt{\overline{\mathcal{E}}} < \varepsilon. \tag{3.46}$$

Therefore, by Lemma 2.2 we have $T_0 = \infty$. In addition, (1.23) is an immediate consequence of (3.45). Moreover, (1.24) and (1.25) follow from (1.23) by standard embedding inequalities, e.g. from (1.23) we have

$$\theta - \theta^* \in L^\infty((0,\infty); L^2(0,1)) \tag{3.47}$$

and

$$\theta_x, \theta_{xt} \in L^2((0,\infty); L^2(0,1)). \tag{3.48}$$

We note that (3.48) implies

$$\theta_x(\cdot, t) \to 0 \ \text{ in } \ L^2(0,1) \ \text{ as } \ t \to \infty. \tag{3.49}$$

Observe that

$$\begin{aligned} [\theta(x,t) - \theta^*]^2 &= 2\int_0^x [\theta(\xi,t) - \theta^*]\theta_x(\xi,t)d\xi \le 2\int_0^1 |\theta(\xi,t) - \theta^*|\, |\theta_x(\xi,t)| d\xi \\ &\le 2\Big(\int_0^1 [\theta(\xi,t) - \theta^*]^2 d\xi\Big)^{\frac{1}{2}}\Big(\int_0^1 \theta_x^2(\xi,t)d\xi\Big)^{\frac{1}{2}} \quad x \in [0,1], t \ge 0. \end{aligned} \tag{3.50}$$

Hence, by (3.47) and (3.49)

$$\theta(\cdot,t) \to \theta^* \text{ uniformly on } [0,1] \text{ as } t \to \infty. \tag{3.51}$$

This completes the proof of Theorem 1.1. ∎

The proofs of Theorem 1.2 and 1.3 are very similar to the proof above. In both cases, however, since we cannot use Poincaré's inequality, we do not obtain a temporal-L^2 estimate for $\theta - \theta^*$ and hence before we proceed with the calculations we divide equations (1.1) and (1.40) by $\hat{e}'(\theta(x,t))$. For the same reason, in Theorem 1.2, for example, we require that (1.29) hold in order to obtain the following estimate:

$$\begin{aligned}
&|\int_0^t \int_0^1 \frac{1}{\hat{e}'(\theta(x,s))}[\theta(x,s)-\theta^*]r(x,s)dxds| \\
&\quad \le \Gamma \int_0^t (\int_0^1 [\theta(x,s)-\theta^*]^2 dx)^{\frac{1}{2}} (\int_0^1 r^2(x,s)dx)^{\frac{1}{2}} \\
&\quad \le \Gamma \sup_{s\in[0,t]} (\int_0^1 [\theta(x,s)-\theta^*]^2 dx)^{\frac{1}{2}} \int_0^t (\int_0^1 r^2(x,s)dx)^{\frac{1}{2}} ds \\
&\quad \le \Gamma \sqrt{\mathcal{E}(t)} \int_0^\infty (\int_0^1 r^2(x,t)dx)^{\frac{1}{2}} dt \quad \forall t \in [0,T_0);
\end{aligned} \tag{3.52}$$

the other terms with which one must be careful can be handled by integration by parts, e.g.

$$
\begin{aligned}
&|\int_0^t \int_0^1 \frac{1}{\hat{e}'(\theta(x,s))}[\theta(x,s)-\theta^*]\int_0^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) \\
&\qquad\qquad - F''(0))\bar{\theta}_{xx}^s(x,z)dzdxds| \\
&= |-\int_0^t \int_0^1 \frac{1}{\hat{e}'(\theta(x,s))}[\theta(x,s)-\theta^*]\frac{\partial}{\partial x}\int_0^\infty a(z)(F''(\bar{\theta}_x^s(x,z)) \\
&\qquad\qquad - F''(0))\theta_x(x,s-z)dzdxds| \\
&= |\int_0^t \int_0^1 (-\frac{\hat{e}''(\theta(x,s))}{\hat{e}'(\theta(x,s))^2}[\theta(x,s)-\theta^*] \qquad\qquad (3.53)\\
&\qquad + \frac{1}{\hat{e}'(\theta(x,s))})\theta_x(x,s)\int_0^s a(z)(F''(\bar{\theta}_x^s(x,z)) \\
&\qquad\qquad - F''(0))\theta_x(x,s-z)dzdxds| \\
&\le \Gamma(\nu(t)+\nu^{k+1}(t))\mathcal{E}(t) \\
&\qquad\qquad t\,\epsilon\,[0,T_0)\,.
\end{aligned}
$$

The argument to show that $\theta(\cdot,t)\to\theta^{**}$ uniformly on $[0,1]$ as $t\to\infty$ for Theorem 1.2, is essentially the same as the argument used to establish an analogous result in Section 3 of [4]: One first observes that standard embedding inequalities yield (1.38) as well as boundedness of θ on $[0,1]\times[0,\infty)$. Hence, every sequence of times tending to infinity has a subsequence on which θ converges uniformly to a constant, namely θ^{**}.

Theorems 1.2 and 1.3 can be proved using an argument in the same spirit as in [12], i.e. instead of taking temporal derivatives of the equation and multiplying by corresponding time derivatives of θ, one can take spatial derivatives of the equation and multiply by appropriate x derivatives of θ. This cannot be done for Theorem 1.1 since we have a term involving $\theta_x(x,t)$ on the right-hand side of (1.1) which would lead to uncontrollable boundary terms.

In the case of nonequilibrium history the argument is essentially the same. The main modification needed arises when we want to make use of an inequality of the form (2.21); we then extend a to $\mathbb{R}$ by zero. To give an indication of where such a modification is needed we consider the analogue of the term treated in (3.26):

$$
\begin{aligned}
&\left| \int_0^t \int_0^1 [\theta(x,s) - \theta^*] \int_{-\infty}^s \theta_{xx}(x,y) \int_{s-y}^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) - F''(0)) dz dy dx ds \right| \\
&\qquad \le \left| \int_0^t \int_0^1 [\theta(x,s) - \theta^*] \int_{-\infty}^0 \theta_{xx}(x,y) \int_{s-y}^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) \right. \\
&\qquad\qquad\qquad \left. - F''(0)) dz dy dx ds \right| \\
&\qquad + \left| \int_0^t \int_0^1 [\theta(x,s) - \theta^*] \int_0^s \theta_{xx}(x,y) \int_{s-y}^\infty a'(z)(F''(\bar{\theta}_x^s(x,z)) \right. \\
&\qquad\qquad\qquad \left. - F''(0)) dz dy dx ds \right| .
\end{aligned}
\tag{3.54}
$$

The second term on the right-hand side of (3.54) can clearly be handled in the same manner as in (3.26) and once a is extended by zero, the first term on the right-hand side of (3.54) can also be treated in the same way.

Remark 3.1: *We note that in order to obtain a priori bounds in the above proof it sufficed to assume that the data satisfy* (1.55) - (1.58). *It is in the proof of local existence that we need the original assumptions on the data* (1.12) - (1.18).

Remark 3.2: *If, for example, in the case of Theorem 1.1 assumption* (1.21) *is replaced with*

$$
\begin{aligned}
\Theta_0 + \int_0^1 \theta_0'''(x)^2 dx + R_0 + \sup_{t \ge 0} \int_0^1 (r_x^2 + r_{xt}^2 + r_{tt}^2)(x,t) dx& \\
+ \int_0^1 r_{xx}^2(x,0) dx + \int_0^\infty \int_0^1 (r_x^2 + r_{xt}^2 + r_{ttt}^2)(x,t) dx \le \delta^2&
\end{aligned}
\tag{3.55}
$$

then one can establish the existence of a unique solution $\theta > 0$ *satisfying* (1.22) - (1.25); *moreover,*

$$
\theta_{xxx}, \theta_{xxt}, \theta_{xtt}, \theta_{ttt} \in L^\infty((0,\infty); L^2(0,1)) \cap L^2((0,\infty); L^2(0,1)),
\tag{3.56}
$$

and

$$\theta_{xx}(\cdot,t), \theta_{xt}(\cdot,t), \theta_{tt}(\cdot,t) \to 0 \quad \textit{uniformly on} \quad [0,1] \tag{3.57}$$

as $t \to \infty$. *The arguments used to establish such a result are similar in spirit to the arguments used to prove Theorem 1.1 except that here there is no need to make use of the entropy inequality* (0.2) *or any other consequence of the thermodynamical restrictions.*

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